2. Phase Plane Analysis

Phase plane analysis is a graphical method for studying second-order systems. This chapter's objective is to gain familiarity of the nonlinear systems through the simple graphical method.

2.1 Concepts of Phase Plane Analysis

2.1.1 Phase portraits
The phase plane method is concerned with the graphical study of second-order autonomous systems described by

$$\dot{x}_1 = f_1(x_1, x_2) \qquad (2.1a)$$
$$\dot{x}_2 = f_2(x_1, x_2) \qquad (2.1b)$$

where
  x_1, x_2 : states of the system
  f_1, f_2 : nonlinear functions of the states

Geometrically, the state space of this system is a plane having x_1, x_2 as coordinates. This plane is called *phase plane*. The solution of (2.1) with time varies from zero to infinity can be represented as a curve in the phase plane. Such a curve is called a *phase plane trajectory*. A family of phase plane trajectories is called a *phase portrait* of a system.

Example 2.1 Phase portrait of a mass-spring system_____

(a)   (b)
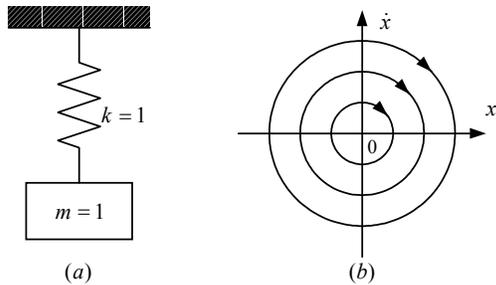
Fig. 2.1 A mass-spring system and its phase portrait

The governing equation of the mass-spring system in Fig. 2.1 is the familiar linear second-order differential equation

$$\ddot{x} + x = 0 \qquad (2.2)$$

Assume that the mass is initially at rest, at length x_0. Then the solution of this equation is

$$x(t) = x_0 \cos(t)$$
$$\dot{x}(t) = -x_0 \sin(t)$$

Eliminating time t from the above equations, we obtain the equation of the trajectories

$$x^2 + \dot{x}^2 = x_0^2$$

This represents a circle in the phase plane. Its plot is given in Fig. 2.1.b.

The nature of the system response corresponding to various initial conditions is directly displayed on the phase plane. In the above example, we can easily see that the system trajectories neither converge to the origin nor diverge to infinity. They simply circle around the origin, indicating the marginal nature of the system's stability.

A major class of second-order systems can be described by the differential equations of the form

$$\ddot{x} = f(x, \dot{x}) \qquad (2.3)$$

In state space form, this dynamics can be represented with $x_1 = x$ and $x_2 = \dot{x}$ as follows

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x_1, x_2)$$

2.1.2 Singular points
A singular point is an equilibrium point in the phase plane. Since an equilibrium point is defined as a point where the system states can stay forever, this implies that $\dot{\mathbf{x}} = \mathbf{0}$, and using (2.1)

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases} \qquad (2.4)$$

For a linear system, there is usually only one singular point although in some cases there can be a set of singular points.

Example 2.2 A nonlinear second-order system_____
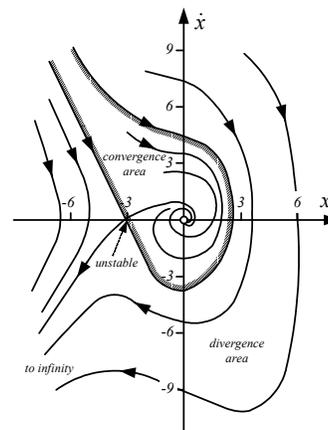
Fig. 2.2 A mass-spring system and its phase portrait

Consider the system $\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$ whose phase portrait is plot in Fig. 2.2.

The system has two singular points, one at $(0,0)$ and the other at $(-3,0)$. The motion patterns of the system trajectories in the vicinity of the two singular points have different natures. The trajectories move towards the point $x = 0$ while moving away from the point $x = -3$.

Why an equilibrium point of a second order system is called a *singular* point ? Let us examine the slope of the phase portrait. The slope of the phase trajectory passing through a point (x_1, x_2) is determined by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \tag{2.5}$$

where f_1, f_2 are assumed to be single valued functions. This implies that the phase trajectories will not intersect. At singular point, however, the value of the slope is 0/0, *i.e.*, the slope is indeterminate. Many trajectories may intersect at such point, as seen from Fig. 2.2. This indeterminacy of the slope accounts for the adjective "singular".

Singular points are very important features in the phase plane. Examining the singular points can reveal a great deal of information about the properties of a system. In fact, the stability of linear systems is uniquely characterized by the nature of their singular points.
Although the phase plane method is developed primarily for second-order systems, it can also be applied to the analysis of first-order systems of the form

$$\dot{x} + f(x) = 0$$

The difference now is that the phase portrait is composed of a single trajectory.

Example 2.3 A first-order system_____

Consider the system $\dot{x} = -4x + x^3$ there are three singular points, defined by $-4x + x^3 = 0$, namely, $x = 0, -2, 2$. The phase portrait of the system consists of a single trajectory, and is shown in Fig. 2.3.
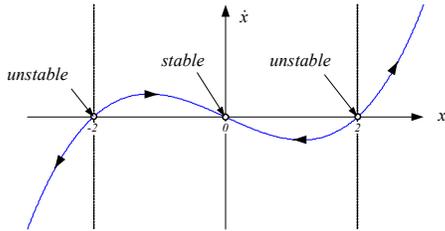
Fig. 2.3 Phase trajectory of a first-order system

The arrows in the figure denote the direction of motion, and whether they point toward the left or the right at a particular point is determined by the sign of $\dot{x}$ at that point. It is seen from the phase portrait of this system that the equilibrium point $x = 0$ is stable, while the other two are unstable.

2.1.3 Symmetry in phase plane portrait

Let us consider the second-order dynamics (2.3): $\ddot{x} = f(x, \dot{x})$. The slope of trajectories in the phase plane is of the form

$$\frac{dx_2}{dx_1} = -\frac{f(x_1, x_2)}{\dot{x}}$$

Since symmetry of the phase portraits also implies symmetry of the slopes (equal in absolute value but opposite in sign), we can identify the following situations:

$$f(x_1, x_2) = f(x_1, -x_2) \quad \Rightarrow \text{ symmetry about the } x_1 \text{ axis.}$$
$$f(x_1, x_2) = -f(x_1, -x_2) \quad \Rightarrow \text{ symmetry about the } x_2 \text{ axis.}$$
$$f(x_1, x_2) = -f(-x_1, -x_2) \Rightarrow \text{ symmetry about the origin.}$$

2.2 Constructing Phase Portraits

There are a number of methods for constructing phase plane trajectories for linear or nonlinear system, such that so-called analytical method, the method of isoclines, the delta method, Lienard's method, and Pell's method.

Analytical method
There are two techniques for generating phase plane portraits analytically. Both technique lead to a functional relation between the two phase variables x_1 and x_2 in the form

$$g(x_1, x_2) = 0 \tag{2.6}$$

where the constant c represents the effects of initial conditions (and, possibly, of external input signals). Plotting this relation in the phase plane for different initial conditions yields a phase portrait.

The first technique involves solving (2.1) for x_1 and x_2 as a function of time t, *i.e.*, $x_1(t) = g_1(t)$ and $x_2(t) = g_2(t)$, and then, eliminating time t from these equations. This technique was already illustrated in example 2.1.

The second technique, on the other hand, involves directly eliminating the time variable, by noting that $\dfrac{dx_2}{dx_1} = \dfrac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$ and then solving this equation for a functional relation between x_1 and x_2. Let us use this technique to solve the mass-spring equation again.

Example 2.4 Mass-spring system_____
By noting that $\ddot{x} = (d\dot{x}/dx)/(dx/dt)$, we can rewrite (2.2) as $\dot{x}\dfrac{d\dot{x}}{dx} + x = 0$. Integration of this equation yields $\dot{x}^2 + x^2 = x_0^2$.

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Most nonlinear systems cannot be easily solved by either of the above two techniques. However, for piece-wise linear systems, an important class of nonlinear systems, this can be conveniently used, as the following example shows.

Example 2.5 A satellite control system_____
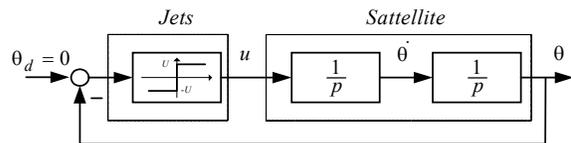
Fig. 2.4 Satellite control system

Fig. 2.4 shows the control system for a simple satellite model. The satellite, depicted in Fig. 2.5.a, is simply a rotational unit inertia controlled by a pair of thrusters, which can provide either a positive constant torque U (positive firing) or negative torque (negative firing). The purpose of the control system is to maintain the satellite antenna at a zero angle by appropriately firing the thrusters.

The mathematical model of the satellite is $\ddot{\theta} = u$, where u is the torque provided by the thrusters and θ is the satellite angle.

Let us examine on the phase plane the behavior of the control system when the thrusters are fired according to the control law

$$u(t) = \begin{cases} -U & if \quad \theta > 0 \\ U & if \quad \theta < 0 \end{cases} \qquad (2.7)$$

which means that the thrusters push in the counterclockwise direction if θ is positive, and vice versa.

As the first step of the phase portrait generation, let us consider the phase portrait when the thrusters provide a positive torque U. The dynamics of the system is $\ddot{\theta} = U$, which implies that $\dot{\theta} \, d\dot{\theta} = U \, d\theta$. Therefore, the phase portrait trajectories are a family of parabolas defined by $\dot{\theta}^2 = 2U\theta + c_1$, where c_1 is constant. The corresponding phase portrait of the system is shown in Fig. 2.5.b.

When the thrusters provide a negative torque $-U$, the phase trajectories are similarly found to be $\dot{\theta}^2 = -2U x + c_1$, with the corresponding phase portrait as shown in Fig. 2.5.c.
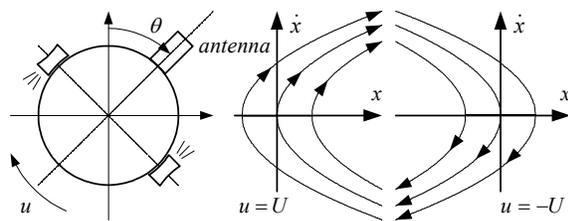
Fig. 2.5 Satellite control using on-off thrusters

The complete phase portrait of the closed-loop control system can be obtained simply by connecting the trajectories on the left half of the phase plane in 2.5.b with those on the right half of the phase plane in 2.5.c, as shown in Fig. 2.6.
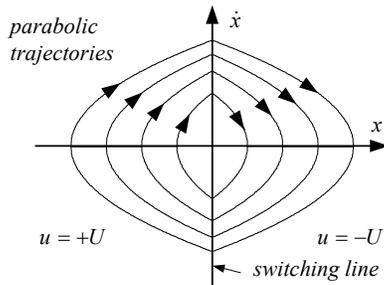
Fig.2.6 Complete phase portrait of the control system

The vertical axis represents a switching line, because the control input and thus the phase trajectories are switched on that line. It is interesting to see that, starting from a nonzero initial angle, the satellite will oscillate in periodic motions under the action of the jets. One can concludes from this phase portrait that the system is marginally stable, similarly to the mass-spring system in Example 2.1. Convergence of the system to the zero angle can be obtained by adding rate feedback.

The method of isoclines (*đường đẳng khuynh*)
The basic idea in this method is that of isoclines. Consider the dynamics in (2.1): $\dot{x}_1 = f_1(x_1, x_2)$ and $\dot{x}_2 = f_2(x_1, x_2)$. At a point (x_1, x_2) in the phase plane, the slope of the tangent to the trajectory can be determined by (2.5). *An isocline is defined to be the locus of the points with a given tangent slope.* An isocline with slope α is thus defined to be

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha$$

This is to say that points on the curve

$$f_2(x_1, x_2) = \alpha \, f_1(x_1, x_2)$$

all have the same tangent slope α.

In the method of isoclines, the phase portrait of a system is generated in two steps. In the first step, a field of directions of tangents to the trajectories is obtained. In the second step, phase plane trajectories are formed from the field of directions.

Let us explain the isocline method on the mass-spring system in (2.2): $\ddot{x} + x = 0$. The slope of the trajectories is easily seen to be

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

Therefore, the isocline equation for a slope α is

$$x_1 + \alpha \, x_2 = 0$$

i.e., a straight line. Along the line, we can draw a lot of short line segments with slope α. By taking α to be different values, a set of isoclines can be drawn, and a field of directions of tangents to trajectories are generated, as shown in Fig. 2.7. To obtain trajectories from the field of directions, we assume that the tangent slopes are locally constant. Therefore, a trajectory starting from any point in the plane can be found by connecting a sequence of line segments.
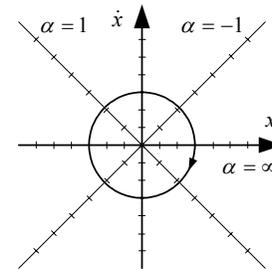
Fig. 2.7 Isoclines for the mass-spring system

Example 2.6 The Van der Pol equation_____

For the Van der Pol equation

$$\ddot{x} + 0.2(x^2 - 1)\dot{x} + x = 0$$

an isocline of slope α is defined by

$$\frac{d\dot{x}}{dx} = -\frac{0.2(x^2 - 1)\dot{x} + x}{\dot{x}} = \alpha$$

Therfore, the points on the curve

$$0.2(x^2 - 1)\dot{x} + x + \alpha\,\dot{x} = 0$$

all have the same slope α .

By taking α of different isoclines can be obtained, as plot in Fig. 2.8.
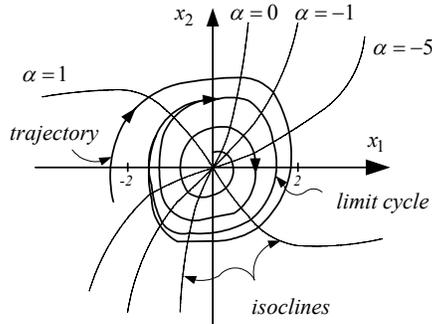
Fig. 2.8 Phase portrait of the Van der Pol equation

Short line segments are drawn on the isoclines to generate a field of tangent directions. The phase portraits can be obtained, as shown in the plot. It is interesting to note that there exists a closed curved in the portrait, and the trajectories starting from both outside and inside converge to this curve. This closed curve corresponds to a limit cycle, as will be discussed further in section 2.5.

2.3 Determining Time from Phase Portraits

Time t does not explicitly appear in the phase plane having x_1 and x_2 as coordinates. We now to describe two techniques for computing time history from phase portrait. Both of techniques involve a step-by step procedure for recovering time.

Obtaining time from $\Delta t \approx \Delta x / \dot{x}$
In a short time Δt , the change of x is approximately

$$\Delta x \approx \dot{x}\,\Delta t \tag{2.8}$$

where $\dot{x}$ is the velocity corresponding to the increment Δx .

From (2.8), the length of time corresponding to the increment Δx is $\Delta t \approx \Delta x / \dot{x}$. This implies that, in order to obtain the time corresponding to the motion from one point to another point along the trajectory, we should divide the corresponding part of the trajectory into a number of small segments (not necessarily equally spaced), find the time associated with each segment, and then add up the results. To obtain the history of states corresponding to a certain initial condition, we simply compute the time t for each point on the phase trajectory, and then plots x with respects to t and $\dot{x}$ with respects to t .

Obtaining time from $t \approx \int (1/\dot{x})\,dx$

Since $\dot{x} = dx / dt$, we can write $dt = dx / \dot{x}$. Therefore,

$$t - t_0 \approx \int_{x_0}^{x} (1/\dot{x})\,dx$$

where x corresponding to time t and x_0 corresponding to time t_0 . This implies that, if we plot a phase plane portrait with new coordinates x and $(1/\dot{x})$, then the area under the resulting curve is the corresponding time interval.

2.4 Phase Plane Analysis of Linear Systems

The general form of a linear second-order system is

$$\dot{x}_1 = a\,x_1 + b\,x_2 \tag{2.9a}$$
$$\dot{x}_2 = c\,x_1 + d\,x_2 \tag{2.9b}$$

Transform these equations into a scalar second-order differential equation in the form $b\dot{x}_2 = bc\,x_1 + d(\dot{x}_1 - a\,x_1)$. Consequently, differentiation of (2.9a) and then substitution of (2.9b) leads to $\ddot{x}_1 = (a+d)\dot{x}_1 + (cb - a\,d)x_1$. Therefore, we will simply consider the second-order linear system described by

$$\ddot{x} + a\,\dot{x} + b\,x = 0 \tag{2.10}$$

To obtain the phase portrait of this linear system, we solve for the time history

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \quad \text{for } \lambda_1 \neq \lambda_2 \tag{2.11a}$$
$$x(t) = k_1 e^{\lambda_1 t} + k_2\,t\,e^{\lambda_2 t} \quad \text{for } \lambda_1 = \lambda_2 \tag{2.11b}$$

whre the constant λ_1, λ_2 are the solutions of the characteristic equation

$$s^2 + as + b = (s - \lambda_1)(s - \lambda_2) = 0$$

The roots λ_1, λ_2 can be explicitly represented as

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

For linear systems described by (2.10), there is only one singular point $(b \neq 0)$, namely the origin. However, the trajectories in the vicinity of this singularity point can display quite different characteristics, depending on the values of a and b . The following cases can occur
- λ_1, λ_2 are both real and have the same sign (+ or -)
- λ_1, λ_2 are both real and have opposite sign
- λ_1, λ_2 are complex conjugates with non-zero real parts
- λ_1, λ_2 are complex conjugates with real parts equal to 0

We now briefly discuss each of the above four cases

Stable or unstable node (Fig. 2.9.a -b)
The first case corresponds to a *node*. A node can be stable or unstable:
$\quad \lambda_1, \lambda_2 < 0$: singularity point is called *stable node*.
$\quad \lambda_1, \lambda_2 > 0$: singularity point is called *unstable node*.
There is no oscillation in the trajectories.

Saddle point (Fig. 2.9.c)
The second case ($\lambda_1 < 0 < \lambda_2$) corresponds to a *saddle point*. Because of the unstable pole λ_2 , almost all of the system trajectories diverge to infinity.
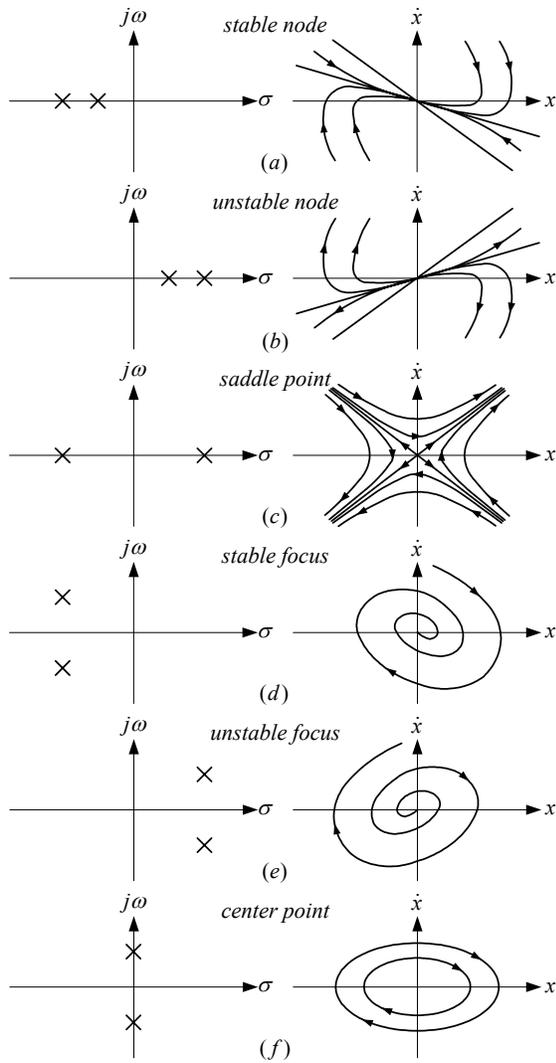
Fig. 2.9 Phase-portraits of linear systems

Stable or unstable locus (Fig. 2.9.d-e)
The third case corresponds to a *focus*.

$\text{Re}(\lambda_1, \lambda_2) < 0 : stable focus$

$\text{Re}(\lambda_1, \lambda_2) > 0 : unstable focus$

Center point (Fig. 2.9.f)
The last case corresponds to a *certain point*. All trajectories are ellipses and the singularity point is the centre of these ellipses.

$\otimes$ *Note that the stability characteristics of linear systems are uniquely determined by the nature of their singularity points. This, however, is not true for nonlinear systems.*

2.5 Phase Plane Analysis of Nonlinear Systems

In discussing the phase plane analysis of nonlinear system, two points should be kept in mind:
- Phase plane analysis of nonlinear systems is related to that of liner systems, because the local behavior of nonlinear systems can be approximated by the behavior of a linear system.
- Nonlinear systems can display much more complicated patterns in the phase plane, such as multiple equilibrium points and limit cycles.

Local behavior of nonlinear systems
If the singular point of interest is not at the origin, by defining the difference between the original state and the singular point as a new set of state variables, we can shift the singular point to the origin. Therefore, without loss of generality, we may simply consider Eq.(2.1) with a singular point at 0. Using Taylor expansion, Eqs. (2.1) can be rewritten in the form

$$\dot{x}_1 = a\,x_1 + b\,x_2 + g_1(x_1, x_2)$$
$$\dot{x}_2 = c\,x_1 + d\,x_2 + g_2(x_1, x_2)$$

where g_1, g_2 contain higher order terms.

In the vicinity of the origin, the higher order terms can be neglected, and therefore, the nonlinear system trajectories essentially satisfy the linearized equation

$$\dot{x}_1 = a\,x_1 + b\,x_2$$
$$\dot{x}_2 = c\,x_1 + d\,x_2$$

As a result, the local behavior of the nonlinear system can be approximated by the patterns shown in Fig. 2.9.

Limit cycle
In the phase plane, *a limit cycle is defied as an isolated closed curve*. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with near by trajectories converging or diverging from it).

Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, we can distinguish three kinds of limit cycles.
- **Stable Limit Cycles**: all trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$ (Fig. 2.10.a).
- **Unstable Limit Cycles**: all trajectories in the vicinity of the limit cycle diverge to it as $t \to \infty$ (Fig. 2.10.b)
- **Semi-Stable Limit Cycles**: some of the trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$ (Fig. 2.10.c)
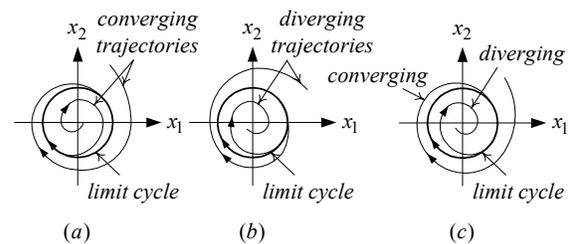
Fig. 2.10 Stable, unstable, and semi-stable limit cycles

Example 2.7 Stable, unstable, and semi-stable limit cycle___

Consider the following nonlinear systems

(a) $\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{cases}$ (2.12)

(b) $\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{cases}$ (2.13)

(c) $\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{cases}$ (2.14)

By introducing a polar coordinates

$$r = \sqrt{x_1^2 + x_2^2} \qquad \theta(t) = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

the dynamics of (2.12) are transformed as

$$\frac{dr}{dt} = -r(r^2 - 1) \qquad \frac{d\theta}{dt} = -1$$

When the state starts on the unicycle, the above equation shows that $\dot{r}(t) = 0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r < 1$, then $\dot{r} > 0$. This implies that the state tends to the circle from inside. When $r > 1$, then $\dot{r} < 0$. This implies that the states tend to the unit circle from outside. Therefore, the unit circle is a stable limit cycle. This can also be concluded by examining the analytical solution of (2.12)

$$r(t) = \frac{1}{\sqrt{1 + c_0 e^{-2t}}} \text{ and } \theta(t) = \theta_0 - t, \text{ where } c_0 = \frac{1}{r_0^2} - 1$$

Similarly, we can find that the system (b) has an unstable limit cycle and system (c) has a semi-stable limit cycle.

2.6 Existence of Limit Cycles

Theorem 2.1 (**Pointcare**) *If a limit cycle exists in the second-order autonomous system (2.1), the N=S+1.*

Where, N represents the number of nodes, centers, and foci enclosed by a limit cycle, S represents the number of enclosed saddle points.

This theorem is sometime called *index theorem*.

Theorem 2.2 (**Pointcare-Bendixson**) *If a trajectory of the second-order autonomous system remains in a finite region* Ω, *then one of the following is true:*
  (a) *the trajectory goes to an equilibrium point*
  (b) *the trajectory tends to an asymptotically stable limit cycle*
  (c) *the trajectory is itself a limit cycle*

Theorem 2.3 (**Bendixson**) *For a nonlinear system (2.1), no limit cycle can exist in the region* Ω *of the phase plane in which* $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ *does not vanish and does not change sign.*

Example 2.8_____

Consider the nonlinear system

$$\dot{x}_1 = g(x_2) + 4x_1 x_2^2$$
$$\dot{x}_2 = h(x_1) + 4x_1^2 x_2$$

Since $\dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2)$, which is always strictly positive (except at the origin), the system does not have any limit cycles any where in the phase plane.