# Module 9: State Feedback Control Design
## Lecture Note 1

The design techniques described in the preceding lectures are based on the transfer function of a system. In this lecture we would discuss the state variable methods of designing controllers.

The advantages of state variable method will be apparent when we design controllers for multi input multi output systems. Moreover, transfer function methods are applicable only for linear time invariant and initially relaxed systems.

## 1 State Feedback Controller

Consider the state-space model of a SISO system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) \end{aligned} \tag{1}$$

where $\mathbf{x}(k) \in R^n$, $u(k)$ and $y(k)$ are scalar. In state feedback design, the states are fedback to the input side to place the closed poles at desired locations.

Regulation Problem: When we want the states to approach zero starting from any arbitrary initial state, the design problem is known as regulation where the internal stability of the system, with desired transients, is achieved. Control input:

$$u(k) = -K\mathbf{x}(k) \tag{2}$$

Tracking Problem: When the output has to track a reference signal, the design problem is known as tracking problem. Control input:

$$u(k) = -K\mathbf{x}(k) + Nr(k)$$

where $r(k)$ is the reference signal.

First we will discuss designing a state feedback control law using pole placement technique for regulation problem.

By substituting the control law (2) in the system state model (1), the closed loop system becomes $\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k)$. If $K$ can be designed such that eigenvalues of $A - BK$ are within the unit circle then the problem of regulation will be solved.

The control problem can thus be defined as: Design a state feedback gain matrix $K$ such that the control law given by equation (2) places poles of the closed loop system $\mathbf{x}(k+1) = (A-BK)\mathbf{x}(k)$ in desired locations.

- A necessary and sufficient condition for arbitrary pole placement is that the pair $(A, B)$ must be controllable.
- Since the states are fedback to the input side, we assume that all the states are measurable.

## 1.1 Designing $K$ by transforming the state model into controllable canonical form

The problem is first solved for the controllable canonical form. Let us denote the controllability matrix by $U_C$ and consider a transformation matrix $T$ as

$$T = U_C W$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$a_i$'s are the coefficients of the characteristic polynomial $|zI - A| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$.

Define a new state vector $\mathbf{x} = T\bar{\mathbf{x}}$. This will transform the system given by (1) into controllable canonical form, as

$$\bar{\mathbf{x}}(k+1) = \bar{A}\bar{\mathbf{x}}(k) + \bar{B}u(k) \tag{3}$$

You should verify that

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \text{ and } \bar{B} = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We first find $\bar{K}$ such that $u(k) = -\bar{K}\bar{\mathbf{x}}(k)$ places poles in desired locations. Since eigenvalues remain unaffected under similarity transformation, $u(k) = -\bar{K}T^{-1}\mathbf{x}(k)$ will also place the poles of the original system in desired locations.

If poles are placed at $z_1, z_2, ...., z_n$, the desired characteristic equation can be expressed as:

$$\begin{aligned} (z-z_1)(z-z_2)\ldots(z-z_n) &= 0 \\ \text{or,} \quad z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n &= 0 \end{aligned} \tag{4}$$

Since the pair $(\bar{A}, \bar{B})$ are in controllable-companion form then, we have

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -(a_n - \bar{k}_1) & -(a_{n-1} - \bar{k}_2) & & \ldots & -(a_1 - \bar{k}_n) \end{bmatrix}$$

Please note that the characteristic equation of both original and canonical form is expressed as: $|zI - A|$=$|zI - \bar{A}|$=$z^n + a_1 z^{n-1} + \ldots + a_n$=0.

The characteristic equation of the closed loop system with $u = -\bar{K}\bar{\mathbf{x}}$ is given as:

$$z^n + (a_1 + \bar{k}_n)z^{n-1} + (a_2 + \bar{k}_{n-1})z^{n-2} + \ldots + (a_n + \bar{k}_1) = 0 \tag{5}$$

Comparing Eqs. (4) and (5), we get

$$\bar{k}_n = \alpha_1 - a_1,\ \bar{k}_{n-1} = \alpha_2 - a_2,\ \bar{k}_1 = \alpha_n - a_n \tag{6}$$

We need to compute the transformation matrix $T$ to find the actual gain matrix $K = \bar{K}T^{-1}$ where $\bar{K} = [\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n]$.

## 1.2 Designing $K$ by Ackermann's Formula

Consider the state-space model of a SISO system given by equation (1). The control input is

$$u(k) = -K\mathbf{x}(k) \tag{7}$$

Thus the closed loop system will be

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k) = \hat{A}\mathbf{x}(k) \tag{8}$$

Desired characteristic Equation:

$$\begin{aligned} |zI - A + BK| = |zI - \hat{A}| &= 0 \\ \text{or,} \quad (z-z_1)(z-z_2)\cdots(z-z_n) &= 0 \\ \text{or,} \quad z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n &= 0 \end{aligned}$$

Using Cayley-Hamilton Theorem

$$\hat{A}^n + \alpha_1 \hat{A}^{n-1} + \ldots + \alpha_{n-1}\hat{A} + \alpha_n I = 0$$

Consider the case when $n = 3$.

$$\begin{aligned}
\hat{A} &= A - BK \\
\hat{A}^2 = (A - BK)^2 = A^2 - ABK - BKA - BKBK &= A^2 - ABK - BK\hat{A} \\
\hat{A}^3 = (A - BK)^3 &= A^3 - A^2BK - ABK\hat{A} - BK\hat{A}^2
\end{aligned}$$

We can then write

$$\begin{aligned}
\alpha_3 I + \alpha_2 \hat{A} + \alpha_1 \hat{A}^2 + \hat{A}^3 &= 0 \\
\text{or,} \quad \alpha_3 I + \alpha_2 (A - BK) + \alpha_1 (A^2 - ABK - BK\hat{A}) + A^3 - A^2BK - ABK\hat{A} - BK\hat{A}^2 &= 0 \\
\text{or,}\ \alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 - \alpha_2 BK - \alpha_1 ABK - \alpha_1 BK\hat{A} - A^2BK - ABK\hat{A} - BK\hat{A}^2 &= 0
\end{aligned}$$

Thus

$$\begin{aligned}
\phi(A) &= B(\alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2) + AB(\alpha_1 K + K\hat{A}) + A^2BK \\
&= \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
&= U_C \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix}
\end{aligned}$$

where $\phi(.)$ is the closed loop characteristic polynomial and $U_C$ is the controllability matrix. Since $U_C$ is nonsingular

$$\begin{aligned}
U_C^{-1}\phi(A) &= \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
\text{or,}\ [0\ \ 0\ \ 1]\, U_C^{-1}\phi(A) &= [0\ \ 0\ \ 1] \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
\text{or,}\ K &= [0\ \ 0\ \ 1]\, U_C^{-1}\phi(A)
\end{aligned}$$

Extending the above for any $n$,

$$K = [0\ \ 0\ \ \ldots\ \ 1]\, U_C^{-1}\phi(A) \qquad \text{where } U_C = [B\ \ AB\ \ A^2B\ \ \ldots\ \ A^{n-1}B]$$

The above equation is popularly known as Ackermann's formula.

**Example 1:** Find out the state feedback gain matrix $K$ for the following system using two different methods such that the closed loop poles are located at 0.5, 0.6 and 0.7.

$$\mathbf{x}(k+1) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Solution:

$$U_C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

The above matrix has rank 3, so the system is controllable.

Open loop characteristic equation:

$$\text{or,}\ \ z^3 + 3z^2 + 2z + 1 = 0$$

Desired characteristic equation:

$$\begin{aligned} (z-0.5)(z-0.6)(z-0.7) &= 0 \\ \text{or,}\ \ z^3 - 1.8z^2 + 1.07z - 0.21 &= 0 \end{aligned}$$

Since the open loop system is already in controllable canonical form, $T = I$.

$$K = [\alpha_3 - a_3 \ \ \alpha_2 - a_2 \ \ \alpha_1 - a_1]$$

where, $\alpha_3 = -0.21,\ \alpha_2 = 1.07,\ \alpha_1 = -1.8$ and $a_3 = 1,\ a_2 = 2,\ a_1 = 3.$ Thus

$$K = [-1.21 \ \ -0.93 \ \ -4.8]$$

Using Ackermann's formula:

$$U_C^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -3 & -1 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \phi(A) &= A^3 - 1.8A^2 + 1.07A - 0.21I \\ &= \begin{bmatrix} -1 & -2 & -3 \\ 3 & 5 & 7 \\ -7 & -11 & -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1.8 \\ 1.8 & 3.6 & 5.4 \\ -5.4 & -9 & -10.6 \end{bmatrix} + \begin{bmatrix} 0 & 1.07 & 0 \\ 0 & 0 & 1.07 \\ -1.07 & -2.14 & -3.21 \end{bmatrix} \\ &\quad + \begin{bmatrix} -0.21 & 0 & 0 \\ 0 & -0.21 & 0 \\ 0 & 0 & -0.21 \end{bmatrix} \\ &= \begin{bmatrix} -1.21 & -0.93 & -4.8 \\ 4.8 & 8.39 & 13.47 \\ -13.47 & -22.14 & -30.02 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} K &= [0\ 0\ 1]U_C^{-1}\phi(A) \\ &= [0\ 0\ 1]\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\phi(A) \\ &= [1\ 0\ 0]\phi(A) = [-1.21 \ \ -0.93 \ \ -4.8] \end{aligned}$$

**Example 2:** Find out the state feedback gain matrix $K$ for the following system by converting the system into controllable canonical form such that the closed loop poles are located at 0.5 and 0.6.

$$\mathbf{x}(k+1) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Solution:

$$U_C = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

The above matrix has rank 2, so the system is controllable.

Open loop characteristic equation:

$$\text{or,}\;\; z^2 + 3z + 2 = 0$$

Desired characteristic equation:

$$\begin{aligned} (z-0.5)(z-0.6) &= 0 \\ \text{or,}\;\; z^2 - 1.1z^2 + 0.3 &= 0 \end{aligned}$$

To convert into controllable canonical form:

$$W = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

The transformation matrix:

$$T = U_C W = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Check:

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now,

$$\alpha_1 = -1.1, \qquad \alpha_2 = 0.3, \qquad a_1 = 3, \qquad a_2 = 2$$

Thus

$$\bar{K} = [\alpha_2 - a_2 \quad \alpha_1 - a_1] = [-1.7 \quad -4.1]$$

We can then write

$$K = \bar{K}T^{-1} = [-1.7 \quad -4.1] \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = [-2.4 \quad -4.1]$$

Taking the initial state to be $\boldsymbol{x}(0) = [2 \quad 1]^T$, the plots for state variables and control variable are shown in Figure 1.

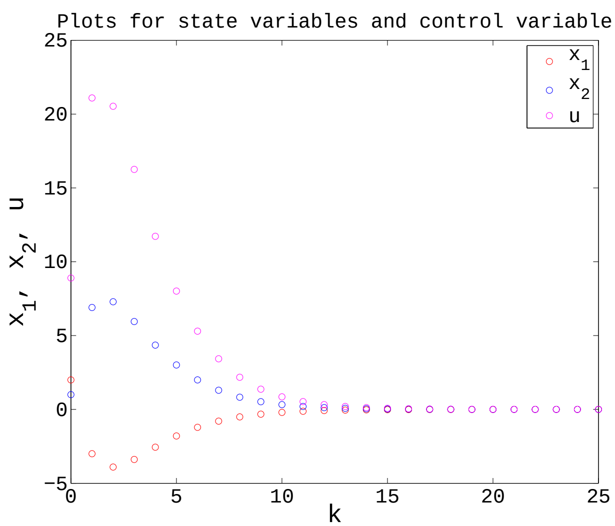


Figure 1: Example 2: Plots for state variables and control variable