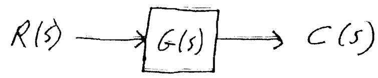
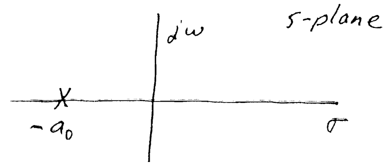
Time Response of First Order Systems

Consider a general first order transfer function (strictly proper)

$$G(s) \quad = \quad \frac{G(s)}{R(s)} = \frac{b_0}{s + a_0}$$





It is common also to write $G(s)$ as

$$G(s) \quad = \quad \frac{K}{\tau s + 1} = \frac{b_0}{s + a_0}$$

i.e.,

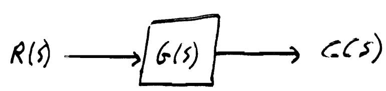
$$a_0 = \frac{1}{\tau} \quad ; \quad b_0 = \frac{K}{\tau}$$

Example:

$$G(s) \quad = \quad \frac{3}{s+2} = \frac{1.5}{0.5s + 1}$$

$$\begin{aligned} a_0 = 2 \quad &, \quad b_0 = 3 \\ \tau = 0.5 \quad &, \quad K = 1.5 \end{aligned}$$

The pole of the system is at $s = -a_0$ or $s = -1/\tau$. τ is called the *time constant*. K is called the *DC-gain* or *steady state gain*.
What is the differential equation corresponding to the input/output system



$$c(s) \quad = \quad \frac{K}{\tau s + 1} R(s)$$

becomes

$$(s + 1/\tau)C(s) \quad = \quad K/\tau R(s)$$

which is equivalent to

$$\dot{c}(t) + \frac{1}{\tau} c(t) \quad = \quad K/\tau r(t)$$

Let us consider the effect of both the input $r(t)$ and the initial condition $c(0)$. Taking Laplace Transform of the differentiation equation — this time including the initial condition yields

$$sC(s) - c(0) + \frac{1}{\tau} c(s) \quad = \quad \frac{K}{\tau} R(s)$$

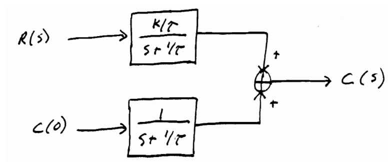
or

$$c(s) \quad = \quad \frac{c(0)}{s+1/\tau} + \frac{K/\tau}{s+1/\tau} R(s).$$

Note that the initial condition could be represented in the differential equation by an input $c(0)\delta(t)$ where δ is the unit impulse function

$$\dot{c}(t) + \frac{1}{\tau} c(t) \quad = \quad \frac{K}{\tau} r(t) + c(0)\delta ?$$

In block diagram from we have



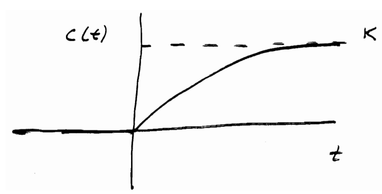
By superposition the response of the system is the sum of the response due to the initial condition alone (the *free response*) and the response due to the input $R(s)$ (the *forced response*).

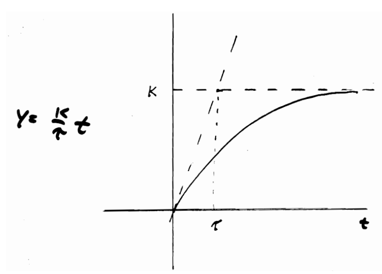
If $R(s) = u(t)$, the unit step function, then the force response (step response) is given (with zero condition) as

$$c(s) \quad = \quad \frac{k/\tau}{s+1/\tau} \cdot \frac{1}{s} = \frac{K}{s} - \frac{K}{s+1/\tau}.$$

In the time domain

$$c(t) \quad = \quad K(1 - e^{-t/\tau})u(t)$$





If the response continued to increase at its initial rate it would reach its steady state value K after τ-seconds.

We see that the forced response is composed of two terms

$$-Ke^{-t/\tau}$$

called the *transient response* and

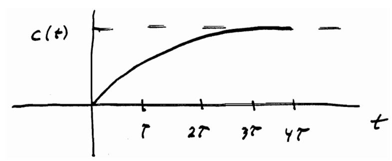
$$K$$

called the *steady state response*.

Then slope of $c(t)$ at $t = 0$ is

$$\left.\frac{d}{dt}c(t)\right|_{t=0} = \left.\frac{K}{\tau}e^{-t/\tau}\right|_{t=0} = \frac{K}{\tau}$$

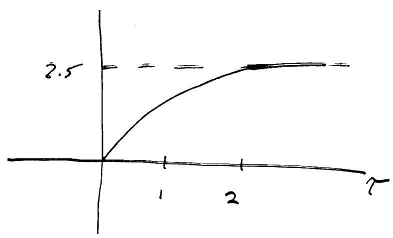
To say it another way the transient response would decay to zero after τ-seconds. In practice we say that the system reaches about 63% ($1 - e^{-1} = .37$) after one time constant and has reached steady state after four time constants.



Example:

$$\begin{aligned} G(s) &= \frac{5}{s+2} \\ &= \frac{2.5}{0.5s+1} \end{aligned}$$

The time constant $\tau = 0.5$ and the steady state value to a unit step input is 2.5.



The classification of system response into

– *forced response*

– *free response*

and

– *transient response*

– *steady state response*

is not limited to first order systems but applies to transfer functions $G(s)$ of any order.

The DC-gain of *any* transfer function is defined as $G(0)$ and is the steady state value of the system to a unit step input, provided that the system has a steady state value. This follows from the final value theorem

$$\begin{aligned} \lim_{t\to\infty} c(t) &= \lim_{s\to 0} sC(s) = \lim_{s\to 0} sG(s)R(s) \\ &= G(0) \text{ if } R(s) = 1/s \end{aligned}$$

provided $sC(s)$ has no poles in the right half plane.

Second Order Systems Consider a second order transfer function

$$G(s) = \frac{c(s)}{R(s)} = \frac{b_0}{s^2 + a_1 s + a_0}.$$

The standard form of this transfer function is

$$G(s) = K \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

omega$_n$ = ω_n is called the *natural frequency* (or undamped natural frequency). *zeta* = ζ is called the *damping ratio.*

The characteristic equation is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \quad = \quad 0$$

which has roots

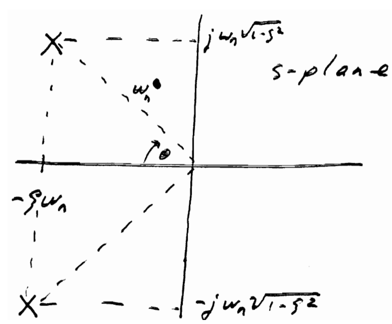
$$\begin{aligned} s &= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \end{aligned}$$

We consider 3 cases

$$\begin{gathered} 0 < \zeta < 1 \\ \zeta = 1 \\ \zeta > 1 \end{gathered}$$

1) if $0 < \zeta < 1$ the system is called *underdamped.* The roots are complex conjugate

$$s \quad = \quad -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$



The unit step response is

$$c(s) \quad = \quad G(s)R(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

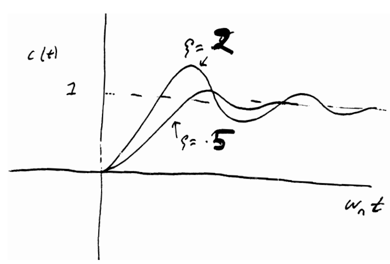
and it can be shown that

$$c(t) \quad = \quad 1 - \frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\beta\omega_n t + \theta)$$

where

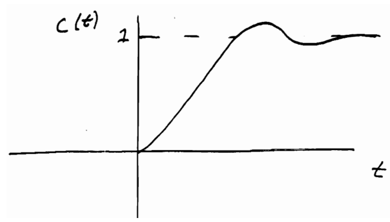
$$\begin{aligned} \beta &= \sqrt{1 - \zeta^2} \\ \theta &= \tan^{-1}(\beta/\zeta) \end{aligned}$$

$\tau = 1/\zeta\omega_n$ is the time constant of the exponentially decaying term. $c(t) \approx 1$ after 4τ.



$$
\begin{aligned}
\text{smaller } \zeta &= \text{more oscillation} \\
&= \text{more overshoot} \\
&= \text{longer time to reach steady state}
\end{aligned}
$$

$\zeta = 0$ is undamped and the oscillations never decay to zero.

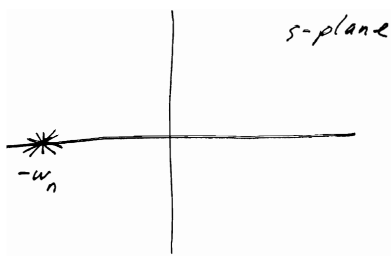


underdamped second-order step response

- decaying oscillation
- overshoot
- constant steady state

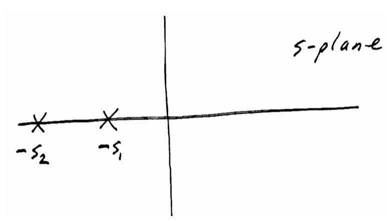
2) $\zeta = 1$ is called *critically damped*. In this case $\sqrt{s^2 - 1} = 0$ and the roots are real and repeated

$$s = -\omega_n$$



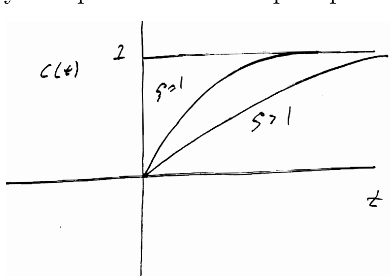
3) $\zeta > 1$ is called *overdamped*. The roots are real and distinct

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



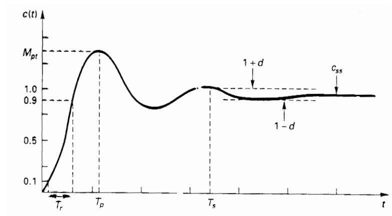
$$
\begin{aligned}
s_1 &= \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \\
s_2 &= \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}
\end{aligned}
$$

In both the overdamped and critically damped cases the step response does not oscillate



$$overshoot$$
$$rise\ time$$
$$settling\ time$$
$$steady\ state\ error$$

are all measures of performance that are used to design control systems.



$$percent\ overshoot$$

$$\frac{M_{pt} - C_{ss}}{C_{ss}} \quad \times \quad 100$$

rise time T_r

is the time required for the step response to rise from 10% to 90% of its final value.

The Settling Time $T_s$is the time required for the response to remain within a certain percent of its final value, typically 2% to 5%. If we use 4 time constants as a measure then

$$\tau_s \quad = \quad 4\tau = 4/\zeta\omega_n$$

These specifications can be used to design ξ, ω. Calculation of percent overshoot. Since the step response (for $\zeta < 1$) is given by

$$c(t) \quad = \quad 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\beta\omega_\tau + \theta)$$

the maximum value M_{pt} can be found from

$$\frac{d}{dt} c(t) \quad = \quad 0 \Rightarrow \sin(\beta\omega_n t) = 0$$

Therefore the maximum occurs at time T_p where $\beta\omega_n T_p = \pi$ or

$$T_p \quad = \quad \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

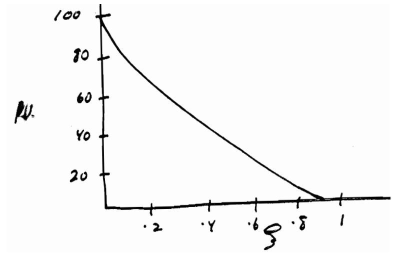
The peak value $M_(pt)$ is calculated from

$$\begin{aligned} C(T_p) &= 1 - \frac{1}{\beta} e^{-\zeta\omega\_nt} \sin(\beta\omega_n t + \sigma)\Big|_{t=T_1} \\ &= 1 = e^{-\zeta\pi/\sqrt{1-\zeta^2}}. \end{aligned}$$

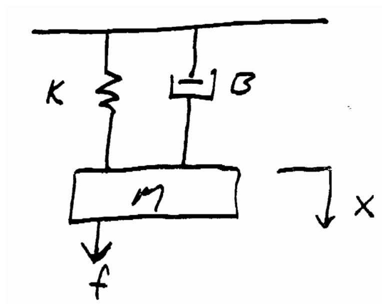
Therefore the percent overshoot is

$$P.O. \quad = \quad e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

and depends only on ζ.



The graph can be used for design.
Example: Suppose you want $M < 20\%$. Then you must design the system so that $\zeta > 0.5$ (approx)
Example:



Suppose M, K are given. How should you choose B (shock absorber) so that the P.O. to a unit step is less than $\sim 4\%$.
Example:

$$\begin{aligned} M &= 100 \\ K &= 1000 \end{aligned}$$

$$100\ddot{x} + B\dot{x} + 1600x = f$$

or

$$\ddot{x} + \frac{B}{100}\dot{x} + 16x = \frac{1}{100}f$$

The characteristic polynomial is

$$S^2 + \frac{B}{100}S + 16 = S^2 + 2\zeta\omega_n S + \omega_n^2.$$

Therefore we want P.O. ¡ 4%, therefore (from the graph) $\zeta > 0.7$ with $\zeta = 0.7$, $\omega_n = 4$ we see that

$$\frac{B}{100} = 2 \cdot 0.7 \cdot 4$$

Therefore

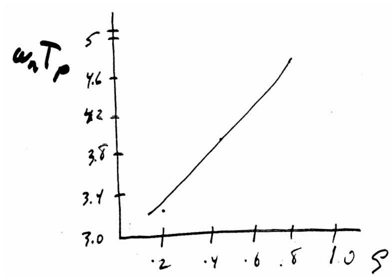
$$B = 560 \left(\frac{nt \cdot sec}{M} \right)$$

Note that T_p is also an indication of rise time. From

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

we have

$$\omega_n T_p \quad = \quad \frac{\pi}{\sqrt{1-\zeta^2}}.$$

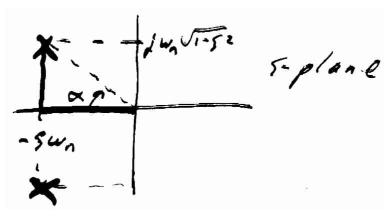


Relation to Pole Location

Since the poles at

$$s \quad = \quad -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

We see that the real part determines the settling time (recall $T_s = 4/\zeta\omega_n$)



$$\alpha \quad = \quad \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \cos^{-1}\zeta$$

Therefore

$$\begin{aligned} P.O. \quad &= \quad e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 \\ &= \quad e^{-\pi/\tan^{\alpha}} \times 100 \end{aligned}$$

Example: Suppose we want $P.O. < 4.32\%$ (which corresponds to $\zeta = \frac{1}{\sqrt{2}} \approx 0.707$) and a settling time $T_s < 2$. Then form $T_s = \frac{2}{\zeta\omega_n} < 2$ we get $\zeta\omega_n > 2$ or $-\zeta\omega_n < -2$ and we have $\alpha^{\angle} \cos^{-1}(0.707) = 45\%$