SYSTEM ANALOGIES

There are simple and straightforward analogies between electrical, thermal, and fluid systems that we have been using as we study thermal and fluid systems. They are detailed in the center column of the table at the end of this handout. The analogies between current, heat flow, and fluid flow are intuitive and can be directly applied; KCL or the like works for all of them. Likewise, the analogies between voltage, temperature and pressure are intuitive and useful. Usually the quantity of interest is the pressure, voltage, or temperature *difference across* some element (although absolute pressure and temperature are well defined quantities, unlike absolute voltage, which is not defined except if some arbitrary reference point, such as the earth, is used). And KVL or something similar works for all three of these *across* variables (v, T, and p).

However, mechanical, chemical, and resource systems don't fit so neatly into this scheme. Chemical and resource systems don't really fit at all. Although the faculty involved in this course have discussed a few possible analogies for chemical and resource systems, none help much or make much sense, and none are in common use. Analogies between mechanical systems and electrical and fluid systems, however, do work well and are in common use. The complication is that there are two ways to make the analogy, both of which work, and both of which have particular advantages and disadvantages.

Mechanical Analogy I: Intuitive

Probably the first analogy that comes to mind between electrical and mechanical systems is that current is kind of like velocity—both are motion of some kind. And voltage is kind of like force—what pushes the current through a resistor. This intuitive analogy is worked out in detail on the right side of the table. Current in an inductor is just like velocity of a mass—both keep going in the absence of any voltage or force, respectively. Dampers are analogous to resistors—the damper resists motion just as a resistor resists current, and both dissipate, rather than store energy. So far the analogies are both intuitive and mathematically neat and tidy.

Springs also have a straightforward analogy in this scheme—they are like capacitors. Running current into a capacitor, building up voltage, is just like having a velocity compressing a spring, building up force. To make the equations analogous, we do have to write them a little differently, however. We wish to express $f = Kx$ in terms of v rather than f since that's the way we have defined the analogy. We can write $f = K\int v dt$, which is now directly analogous to $v = \frac{1}{C}\int i dt$. Thus, we see that K is analogous to $\frac{1}{C}$, which makes sense, because with a given current for a given time, a smaller capacitor will build up more voltage, whereas with a given velocity for a given time, a smaller spring will build up less force. A more common way to write the capacitor law is $i = \boldsymbol{C}\frac{dv}{dt}$; the direct mechanical analogy is then $v = \frac{\mathbf{1}}{\boldsymbol{K}}\frac{df}{dt}$.

Impedances for electrical systems are defined as, for some element or subcircuit x, $Z_x(s) = V_x(s)/I_x(s)$. According to this analogy, mechanical impedance should be defined as $Z_x(s) = F_x(s)/V_x(s)$. That makes a lot of sense. If you have to push harder for the same velocity, you'd call that a higher impedance. If you take the Laplace transform of the element equations for masses and inductors, you get Ms and Ls; both have impedance that increase with frequency (because it is hard to quickly change the velocity of a mass or the quickly change the current in an inductor). Springs and capacitors have impedances of K/s and 1/(sC), respectively. Both of these increase at low frequency. Again this makes sense. If you try to keep putting a steady current into a capacitor for a long time (low frequency), you will end up with a pretty big voltage. Likewise, if you keep compressing a spring at a steady velocity for a long time, you are going to end up with a pretty big force.

With all of these analogies working out so nicely, you might wonder what could possibly be wrong with this approach. The problem shows up as soon as you start to try to use it to model a system, so let's try an example. Consider the example of a train car with a damper as a coupler to an engine, as shown in Fig. 1.

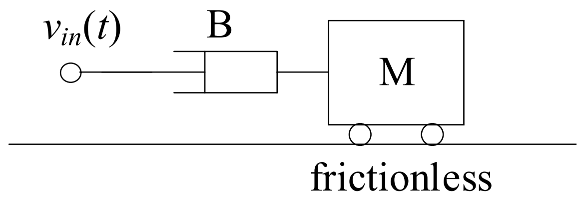

Fig. 1. Mechanical system.

The input $v_{in}(t)$ is a step function—zero until the engine couples, and then at a constant velocity V_0 thereafter. The velocity of the car doesn't change instantaneously, so it starts at zero, and follows a saturating exponential up to V_0, with a time constant M/B. The force of the damper on the car is equal to the force of the engine on the damper. Both forces are initially equal to $B\ V_0$, and they decay exponentially to zero, at which point the car is moving at constant velocity on frictionless rollers and needs no more force. (Exercise for the reader: if it is not clear why the above results are true, draw a free body diagram, write a system equation, and use first-order linear system solution methods to derive them)

If we wanted to use the intuitive analogy we'd first think to try the following electrical equivalent, replacing the mass with an inductor, the damper with a resistor, and the input velocity step function with a current source step function, and putting the elements in the same configuration as above. The result is shown in Fig. 2. The element values are just the same as the element values in the mechanical system (the $v_{in}(t)$ there refers to velocity not voltage—we've set the current equal to velocity for the analogy).

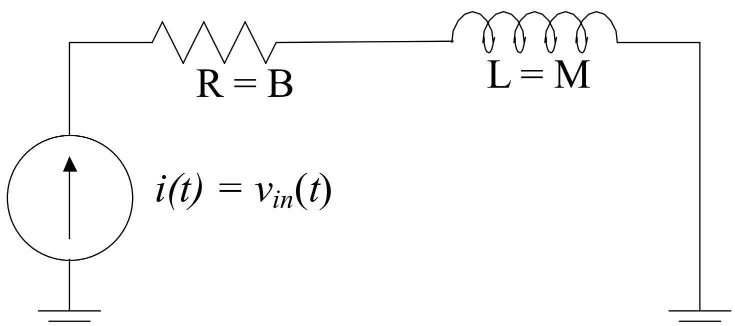

Fig. 2. An incorrect attempt to model the system in Fig. 1 with an electrical system.

This model looks fine at first glance, but consider what happens when $i(t)$ is a step function. The current through the whole series path must be equal, so the current through the inductor must immediately steps up to the source current value. But the current in an inductor can't jump instantaneously without an infinite voltage. So this system requires an infinite voltage, corresponding to infinite force in the mechanical system, and the car accelerates instantaneously. This is not all the behavior of the mechanical system! Suddenly our analogy has broken down and isn't giving sensible results at all anymore.

The problem is that although we have considered the element laws and their analogies carefully, we forgot to consider the connection laws (e.g. KCL and KVL). In the mechanical system, the forces (analogous to voltage) must be the same at both ends of the damper, but the velocities of the mass and the engine (analogous to currents) can be different. By drawing a series circuit, we've set it up so that the current (analogous to velocity) must be the same in both the source and the mass, unlike in the mechanical system. And we've allowed the voltages (analogous to force) to be different where the mass connects to the damper and where the mass connects to the input. So we have all the constraints imposed by the connections wrong!

We can fix it by redrawing as shown in Fig. 3. Now the voltage across the inductor is the same as the voltage across the input, corresponding to the force on the mass being the same as the force on applied by the engine (through the damper, or in the electrical model, across the resistor). The velocity at which the damper is compressed is the difference between the engine velocity and the mass velocity. Analogously, the current in the resistor is the difference between the source current and the current in the inductor.

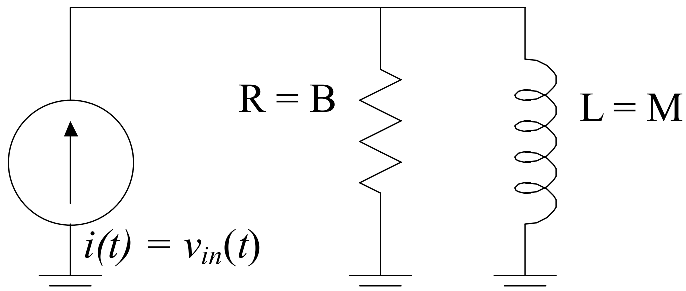

Fig. 3. One correct way to model the system in Fig. 1 with an electrical system. This is using the intuitive analogy.

The behavior of the system in Fig. 3 is in fact exactly the same as the behavior of the system in Fig. 1. When the input (current source) steps up, the input current doesn't match the inductor current, which is still zero, so initially the difference goes through the resistor, and a voltage ($v = iR$) is applied across the inductor. The current in the inductor then grows until it matches the input current, as the voltage decays to zero. The time constant is $L/R = M/B$. The description can be repeated for the mechanical system, by substituting the word velocity for current and force for voltage, and making only a few other cosmetic changes:
When the input (velocity source) steps up, the input velocity doesn't match the mass velocity, which is still zero, so initially the difference is taken up by the damper, and a force ($f = bv$) is applied to the mass. The velocity of the mass then grows until it matches the input velocity, as the force decays to zero.

What we have done in going from the arrangement in Fig.1 and 2 to the new arrangement in Fig. 3 is to swap things around so that everything that was in series (in the original mechanical topology) is now in parallel. If there had been anything in parallel in the original system, we would have drawn the analogy with those elements in series. For example, if the original system had had two identical dampers in parallel, that would have been like a damper with twice the force for a given velocity; a damper with twice the value of B. We could the draw the analogy with two resistors in series to also give twice the value.

This process of swapping series and parallel configurations is called taking the dual of a network. The disadvantage of the intuitive analogy set for mechanical systems is that the analogous electrical and other systems don't have the same topology. But this is only an inconvenience. One can draw the electrical analogy as a dual, and get a model that matches the behavior correctly.

Mechanical Analogy II: Through/Across

In order to avoid the topological confusion that arose in the intuitive analogy, we can start by considering the interconnection laws. The forces at a (massless) node must sum to zero, just as the currents at a (capacitance-less) node must sum to zero. The force *through* a set of series elements (except masses) must be the same, just as the current through a set of series elements must be the same. Thus, if we consider current analogous to force, we may do better. That would leave voltage analogous to velocity.

Considering voltage analogous to velocity has the immediate advantage that the same letter is used for both, but it also has the advantage that both are *across variables*. It is the velocity difference between the ends of the damper that matters, just as it is the voltage difference across a resistor that matters. As a result of the fact that both are across variables, both obey KVL. Kirchoff's Velocity Law is not usually discussed (and when it is, it is not given that name), but it is equivalent to the kinematic equations we would write down for the mechanical system in Fig. 1. We start at "ground"(the stationary reference frame) and go around the loop counter clockwise to add up the velocity of the mass (relative to the stationary reference frame), the velocity difference across the damper, and the velocity difference between ground and the input, and the sum must equal zero:

$$(v_m - 0) + (v_{in} - v_m) + (0 - v_{in}) = 0 \,.$$

If we follow through with these analogies for variables, for a resistor, we want an analogous element with $v = f R_{\text{mech}}$. That element is again a damper, but the values are reciprocals; $R = 1/B$. For a capacitor we want an analogy to $i = \boldsymbol{C}\frac{dv}{dt}$, which should be of the form, $f = C_{mech}\frac{dv}{dt}$. Such a relationship is quite familiar if we use the analogy $M = C_{\text{mech}}$. For an inductor, the analogy again involves the reciprocal, $v = \frac{\boldsymbol{1}}{\boldsymbol{K}}\frac{df}{dt}$, such that the analogy is $L_{\text{mech}} = 1/K$. Gaining an intuitive understanding of these analogies is more difficult for most people, but mathematically, they work out just fine. And they are easier to apply, because the topology does not change. The details are on the left side of the table. Note that everything works out just as nicely here as it does on the right half of the chart. The energy storage and energy dissipation formulas all work out to be exactly analogous.

This sounds good, but our last analogy system crashed only when we tried to run an example, so let's consider the same system in Fig. 1. Since we have carefully considered which are through and across variables, and made the new analogy follow them rigorously, the topology will be preserved. We can make an analogy by just transcribing the same topology with the new element analogies. The topology is like Fig. 2, but now we have different elements and values, as shown in Fig. 4.

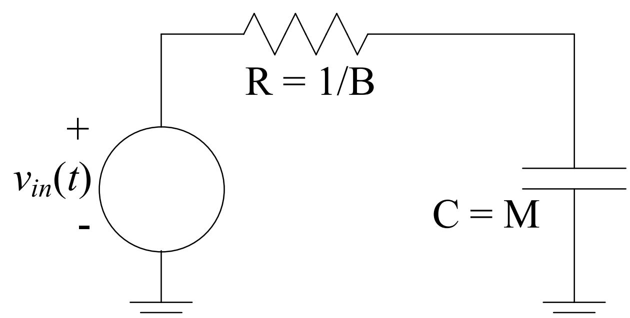

Fig. 4. A second correct way to model the system in Fig. 1 with an electrical system. This is using the through/across analogy.

A rigorous analysis of the behavior of this system, to check that it matches the behavior of the mechanical system in Fig. 1, is left as an exercise. However, consider the description of behavior under Fig. 3, which matched nearly word-for-word a description of the mechanical system. If we try that for this system, it works just fine:

When the input (voltage source) steps up, the input voltage doesn't match the capacitor voltage, which is still zero, so initially the voltage difference appears across the resistor, and a current ($i = v/R$) goes through the resistor to the capacitor. The voltage across the capacitor then grows until it matches the input voltage, as the current decays to zero. The time constant is RC = M/B.

The main advantage of this analogy set is that it preserves the topology. Another advantage is that it is the analogy that is implemented by many electric motors. A permanent-magnet dc motor produces a torque that is approximately proportional to current and produces a voltage that is approximately proportional to angular velocity. This can make modeling mixed mechanical-electrical systems easier.

The main disadvantage of this analogy is that it is less intuitive than our original analogy. A further disadvantage is that it would result in a counterintuitive definition of impedance. The analogy to impedance that is universally used is the one discussed in the intuitive analogy and listed in the corresponding section of the table (on the right).

Mechanical Analogies: Conclusions

There are two fully valid ways of modeling mechanical systems with electrical systems, or drawing analogies between the two types of systems. One method uses an intuitive analogy between current and velocity (both are motion) and voltage force (both are “push”). This works, but it requires constructing a model in a different configuration, with series connections replaced by parallel and vice versa. The second method uses voltage as the analogy for velocity and current as the analogy for force. Although that is counterintuitive, voltage and velocity are both across variables and current and force are both through variables, and thus this analogy leads to analogous systems that have the same topology.

Some books present one of these as the correct analogy; some present the other. Knowing that both are valid puts you one step ahead of many textbook authors.

Notes On a Few Anomalies

- In the energy-storage and energy dissipation equations, everything works analogously except for thermal systems. That’s because one of the variables, heat, is already energy. So a thermal resistor doesn’t dissipate energy, but rather transfers it.
- Masses, thermal capacitors, and most fluid capacitors all have the characteristic that one end must be “grounded”, meaning that the across variable must have one end at zero. That means that you can construct an electrical analogy for any of those systems (just by grounding one end of the appropriate element), but you can’t construct thermal or mechanical analogies for every electrical network. Thus, electrical networks work best as a general-purpose toolkit for modeling any system. Fluid systems could in theory also be used, but the concept of lumped-element fluid systems is relatively rarely used, and inertance is not familiar to enough people for this analogy to be readily and widely understood.

System Analogies

Topology-Preserving Set (book's analogy) — Description, Trans Mech, Rot Mech, Electrical, Thermal, Fluid

Intuitive Analogy Set — Electrical, Thermal, Fluid, Trans Mech, Rot Mech, Description

⇔ intuitive stretch (between Rot Mech and Electrical)

⇔ topology change (between Fluid and Trans Mech)

Description	Trans Mech	Rot Mech	Electrical	Thermal	Fluid	Trans Mech	Rot Mech	Description
"through" variable	$\boldsymbol{f}$ (force)	τ (torque)	$\boldsymbol{i}$ (current)	ϕ (heat flux)	$\boldsymbol{q}$ (flow)	$\boldsymbol{v}$ (velocity)	ω (angular velocity)	Motion
"across" variable	$\boldsymbol{v}$ (velocity)	ω (angular velocity)	$\boldsymbol{v}$ (voltage)	$\boldsymbol{T}, \theta$ (temper-ature)	$\boldsymbol{p}$ (pressure)	$\boldsymbol{f}$ (force)	τ (torque)	Push (force)
Dissipative element	$v = \frac{1}{\boldsymbol{B}} f$	$\omega = \frac{1}{\boldsymbol{B_r}} \tau$	$v = i\boldsymbol{R}$	$\theta = \phi \boldsymbol{R}$	$p = q\boldsymbol{R}$	$f = v\boldsymbol{B}$	$\tau = \omega \boldsymbol{B_r}$	Dissipative element
Dissipation	$f^2 \frac{1}{\boldsymbol{B}} = \frac{v^2}{1/\boldsymbol{B}}$	$\tau^2 \frac{1}{\boldsymbol{B}} = \frac{\omega^2}{1/\boldsymbol{B}}$	$i^2\boldsymbol{R} = v^2/\boldsymbol{R}$	N/A	$q^2\boldsymbol{R} = p^2/\boldsymbol{R}$	$v^2\boldsymbol{B} = f^2/\boldsymbol{B}$	$\omega^2\boldsymbol{B_r} = \tau^2/\boldsymbol{B_r}$	Dissipation
Through-variable storage element	$v = \frac{1}{\boldsymbol{K}}\frac{df}{dt}$ or $\int v\,dt = \frac{1}{\boldsymbol{K}} f$	$\omega = \frac{1}{\boldsymbol{K_r}}\frac{d\tau}{dt}$ or $\int \omega\,dt = \frac{1}{\boldsymbol{K_r}}\tau$	$v = \boldsymbol{L}\frac{di}{dt}$	N/A	$p = \boldsymbol{I}\frac{dq}{dt}$	$f = \boldsymbol{M}\frac{dv}{dt}$ (one end must be "grounded")	$\tau = \boldsymbol{J}\frac{d\omega}{dt}$ (one end must be "grounded")	Motion storage element
Energy	$E = \frac{1}{2}\frac{1}{\boldsymbol{K}} f^2$	$E = \frac{1}{2}\frac{1}{\boldsymbol{K_r}}\tau^2$	$E = ½\boldsymbol{L}i^2$		$E = ½\boldsymbol{I}q^2$	$E = ½\boldsymbol{M}v^2$	$E = ½\boldsymbol{J}\omega^2$	Energy
Impedance	Standard definition is at right		$V(s)=I(s)\boldsymbol{Ls}$		$P(s) = Q(s)\boldsymbol{Is}$	$F(s) = V(s)\boldsymbol{Ms}$	$T(s) = \Omega(s)\boldsymbol{Js}$	Impedance
Across-variable storage element	$f = \boldsymbol{M}\frac{dv}{dt}$ (one end must be "grounded")	$\tau = \boldsymbol{J}\frac{d\omega}{dt}$	$i = \boldsymbol{C}\frac{dv}{dt}$	$\phi = \boldsymbol{C}\frac{d\theta}{dt}$ (one end must be "grounded")	$q = \boldsymbol{C}\frac{dp}{dt}$ (one end usually "grounded")	$v = \frac{1}{\boldsymbol{K}}\frac{df}{dt}$ or $\int v\,dt = \frac{1}{\boldsymbol{K}} f$	$\omega = \frac{1}{\boldsymbol{K_r}}\frac{d\tau}{dt}$ or $\int \omega\,dt = \frac{1}{\boldsymbol{K_r}}\tau$	Push (force) storage element
Energy	$E = ½\boldsymbol{M}v^2$	$E = ½\boldsymbol{J}\omega^2$	$E = ½\boldsymbol{C}v^2$	*$E = \boldsymbol{C}T$ (not analogous)*	$E = ½\boldsymbol{C}\,p^2$	$E = \frac{1}{2}\frac{1}{\boldsymbol{K}} f^2$	$E = \frac{1}{2}\frac{1}{\boldsymbol{K_r}}\tau^2$	Energy
Impedance	The standard definition of mechanical impedance is the one on the right, based on the intuitive analogy.		$V(s) = I(s)\frac{1}{\boldsymbol{sC}}$	$\Theta(s) = \Phi(s)\frac{1}{\boldsymbol{sC}}$	$P(s) = Q(s)\frac{1}{\boldsymbol{sC}}$	$F(s) = V(s)\frac{\boldsymbol{K}}{\boldsymbol{s}}$	$T(s) = \Omega(s)\frac{\boldsymbol{K_r}}{\boldsymbol{s}}$	Impedance

Note: None of the across variables are written as, for example, $(\theta_1 - \theta_2) = \phi R$. That is because the variable *for a particular element* is implicitly the difference across it for any across variable.