University of Cagliari
Dept of Electrical and Electronic Eng.

Describing Function analysis *of* nonlinear systems

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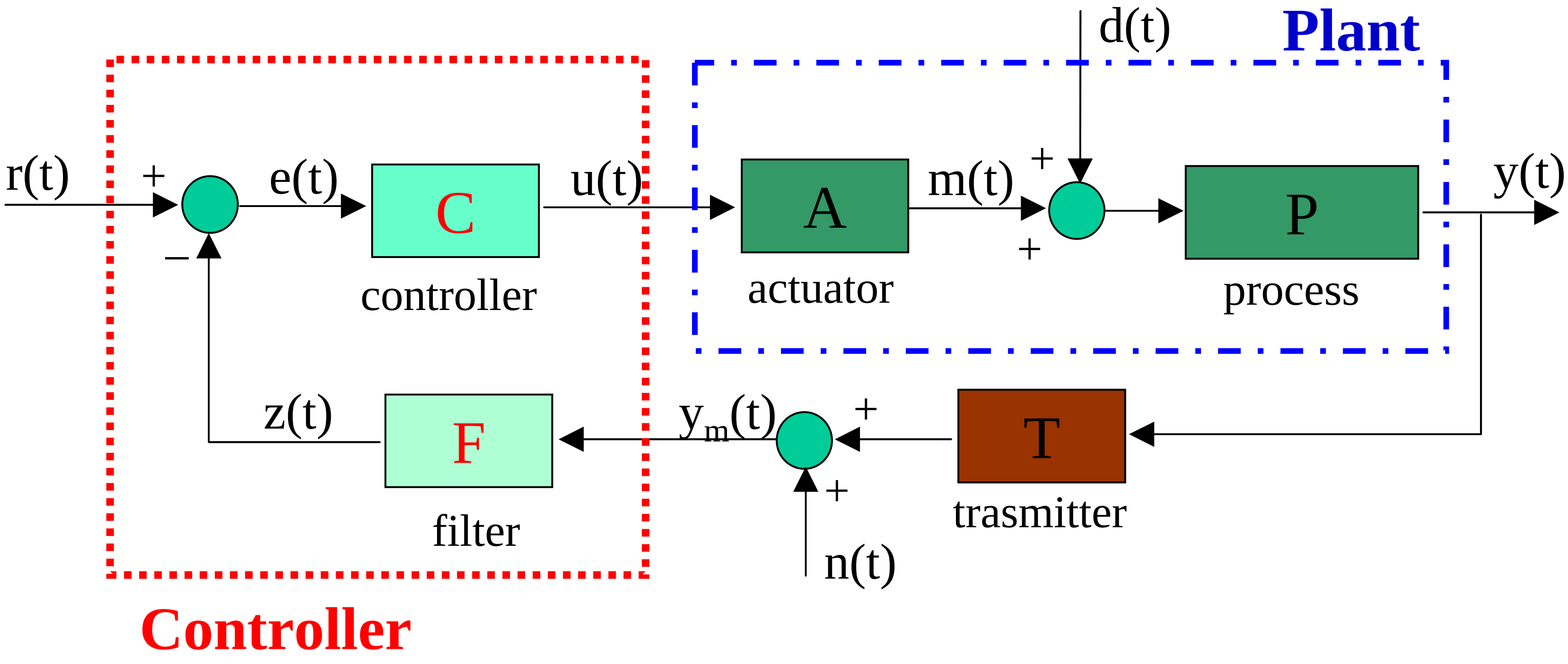
System definition and problem statement



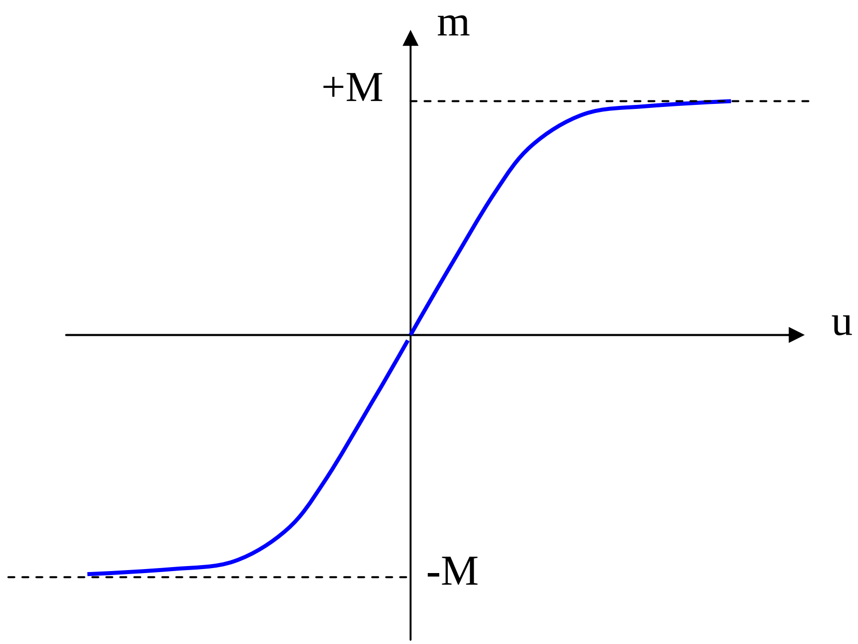
r: reference variable
e: error signal
u: control variable
m: manipulated input
y: controlled variable

y_m: measure of the output
z: feedback signal
d: disturbance
n: measurement noise

System definition and problem statement

In many cases the system presents a nonlinear phenomenon which is fully characterised by its static characteristics, i.e., its dynamics can be neglected

- Saturated actuators
- Relay control
- Gears backlash
- Hysteresis in magnetic materials
- Dead zone in electro-mechanical systems



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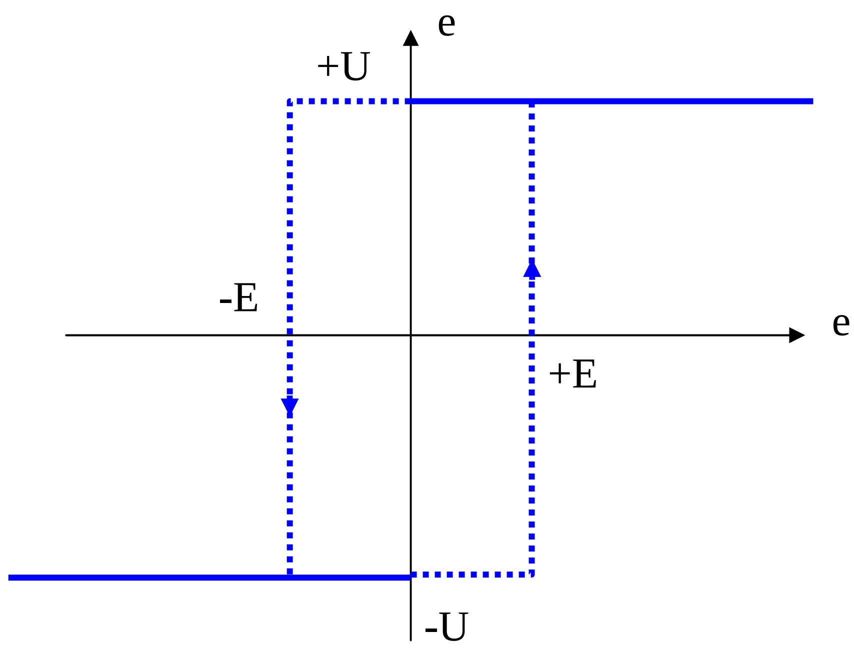

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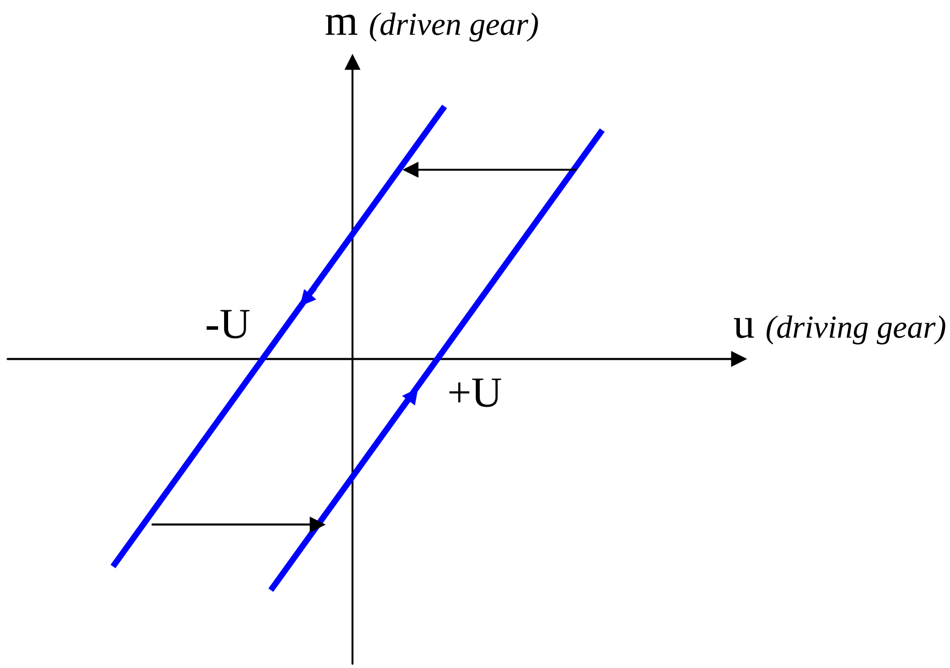

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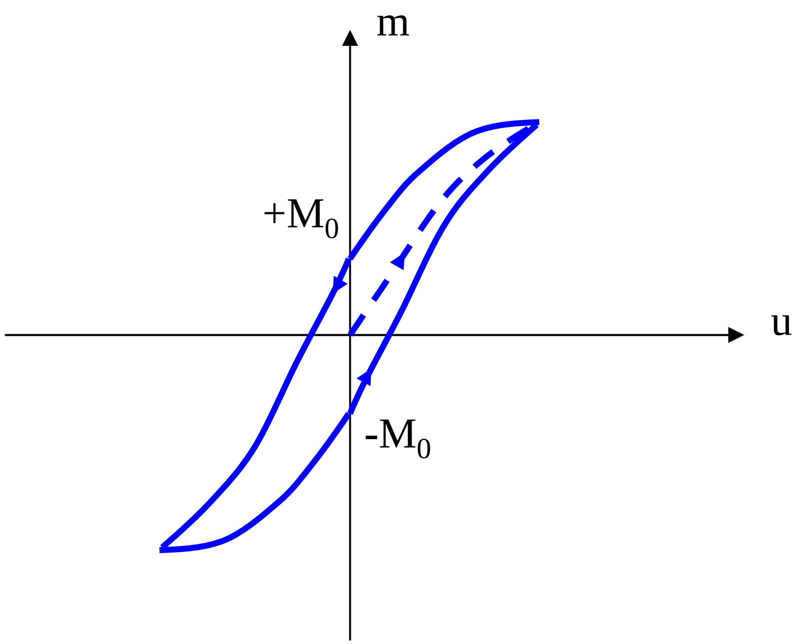

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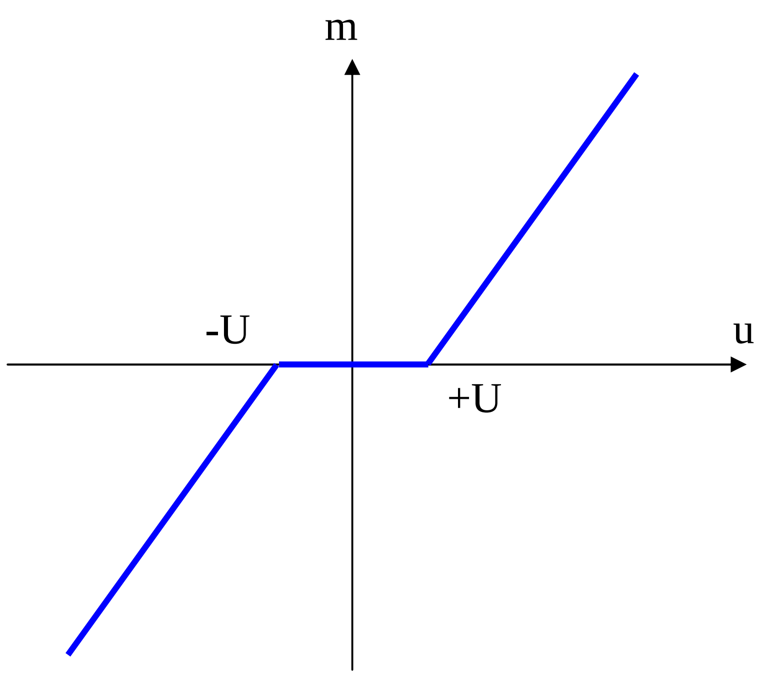
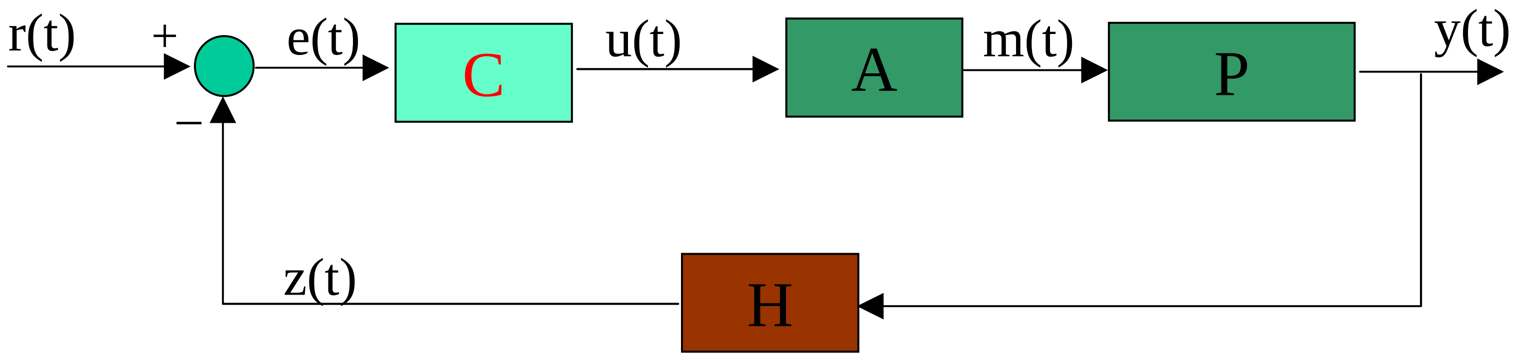


System definition and problem statement

Nonlinearities are not always a drawback, they can also a have a "*stabilising*" effect



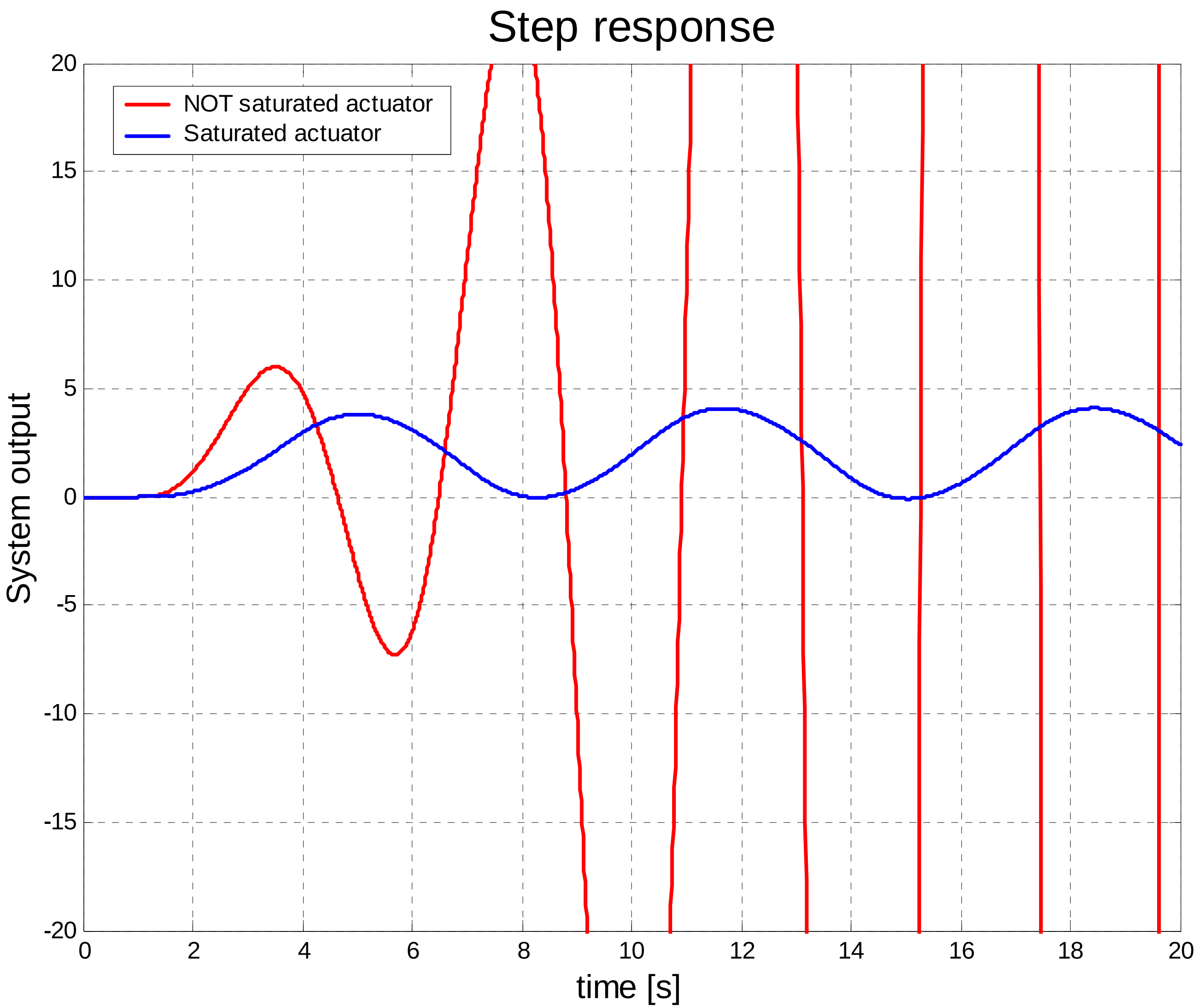
$$P(s) = \frac{1}{s(1 + 1.4s + s^2)}$$

$$H(s) = \frac{0.5}{(1 + 0.1s)}$$

$$C(s) = 10$$

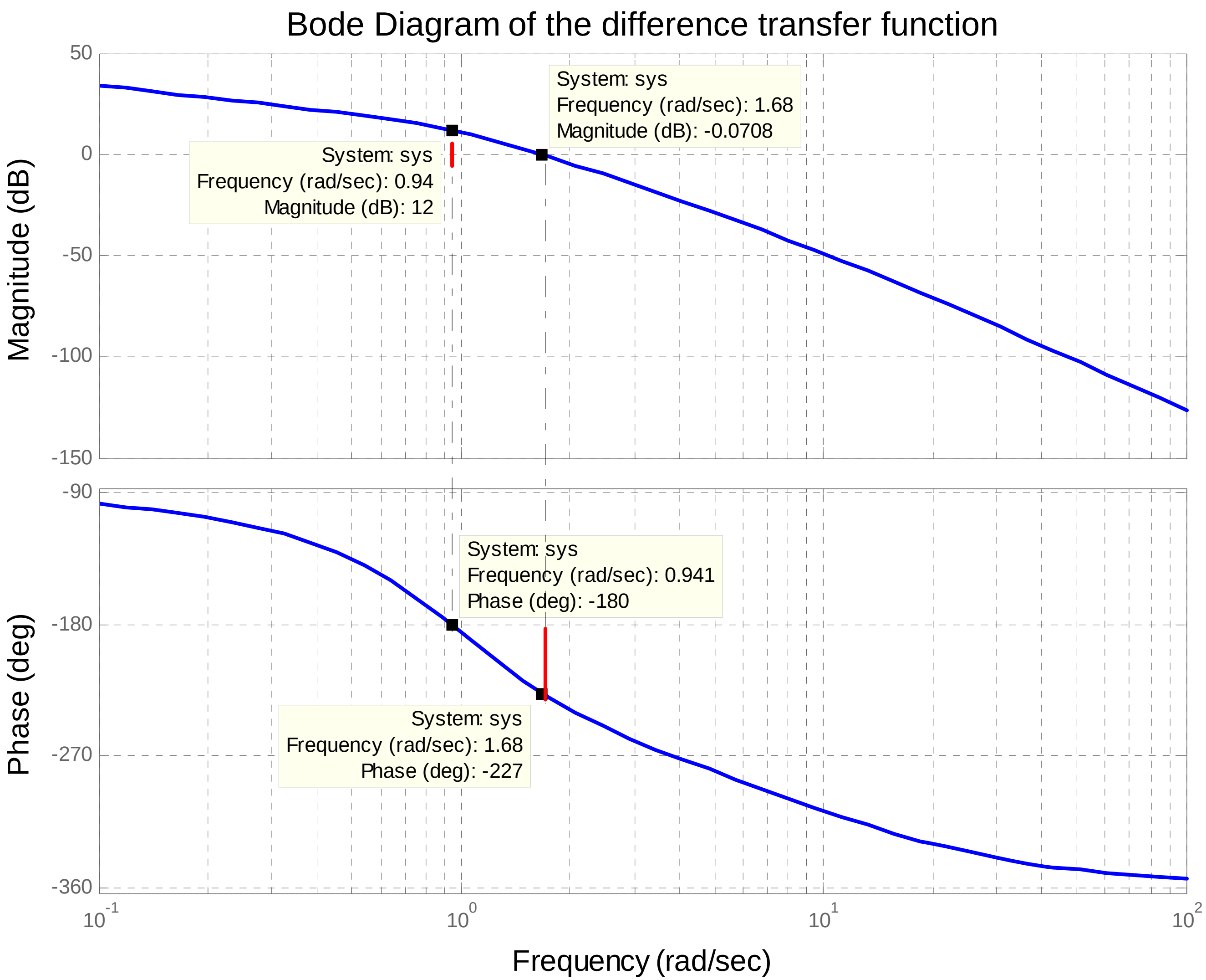
$$A(s) = \begin{cases} 1 \\ sat(\pm 2) \end{cases}$$

System definition and problem statement



The saturated system oscillates but does not diverge

System definition and problem statement

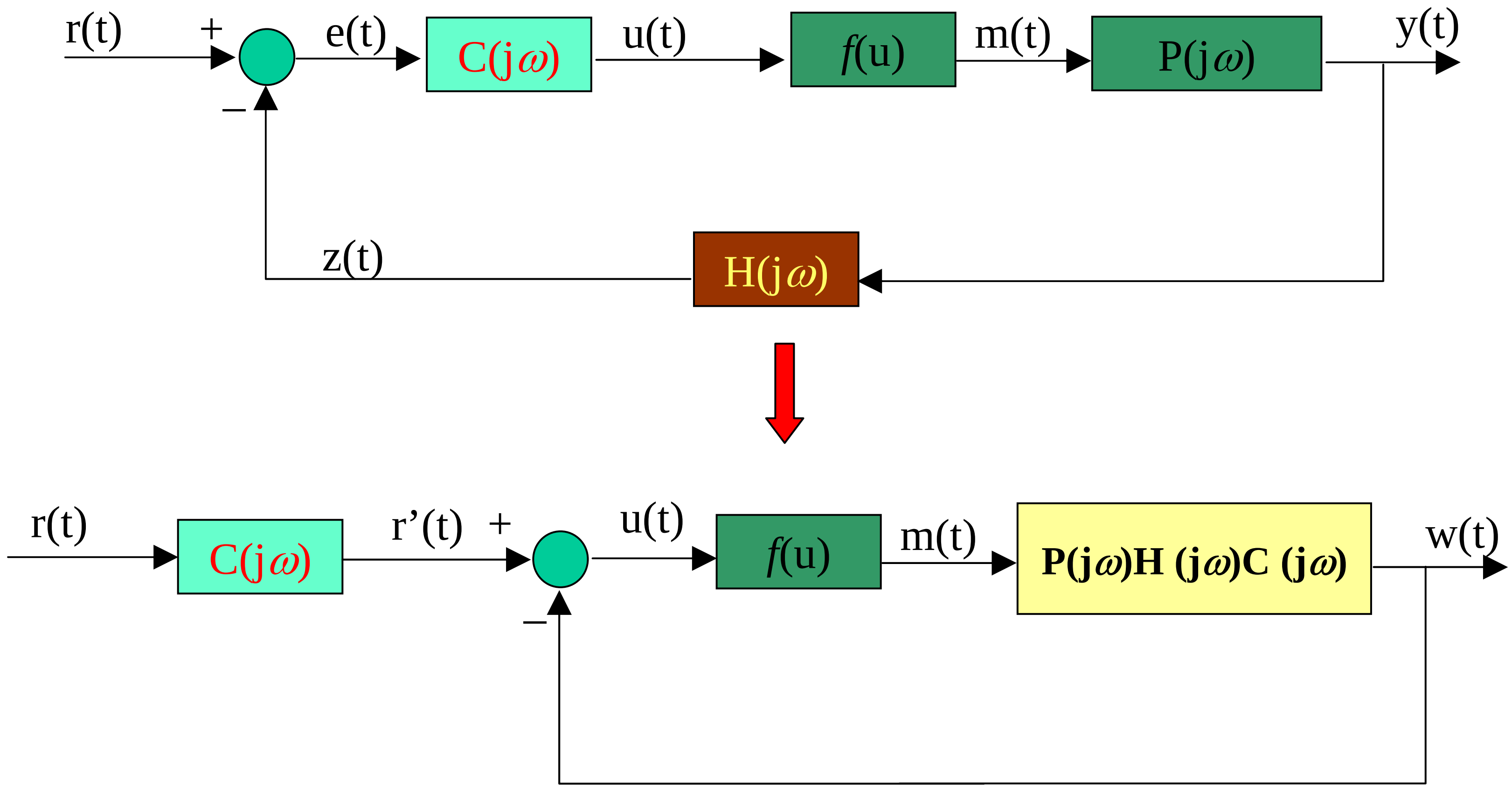


The NOT saturated system is not stable since its stability margin are negatives

- $m_g = -12\ dB$
- $m_\varphi = -\ 47\ deg$

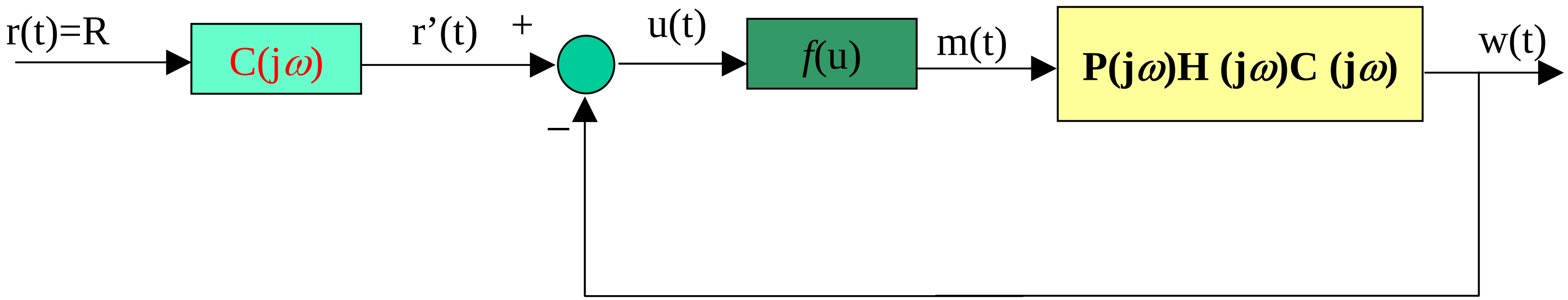
System definition and problem statement

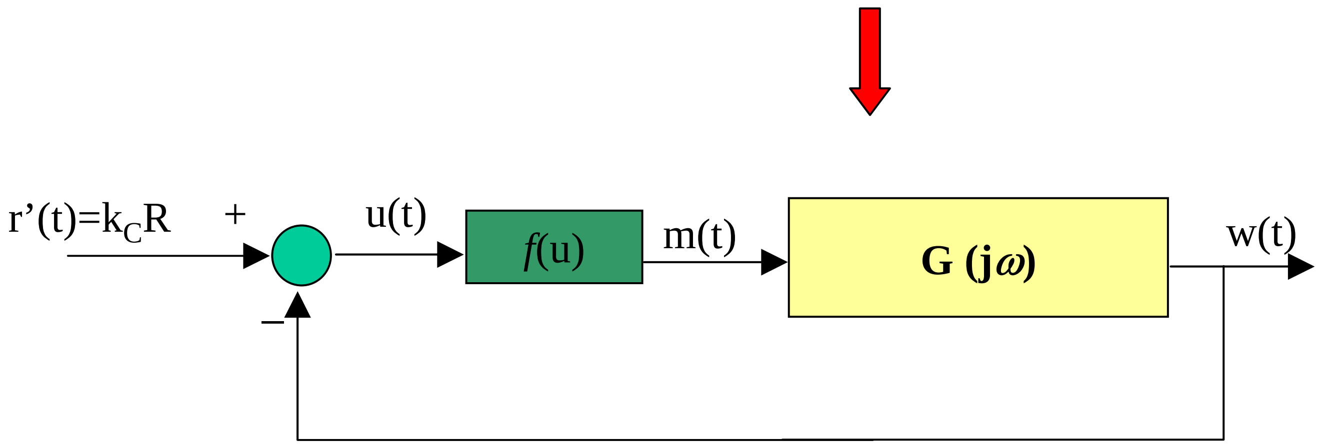
Many systems can be reduced to a simplified form in which the all linear dynamics is concentrated in a unique block and the static non linear characteristics is represented by a separate block



System definition and problem statement

If constant (or very slowly varying) reference signals are considered, under some conditions it is possible to separate the low-frequency, almost static, behaviour defined by a working nominal condition and a high-frequency behaviour due to small variations around such a working point





$$u(t) = u_0 + \Delta u(t)$$

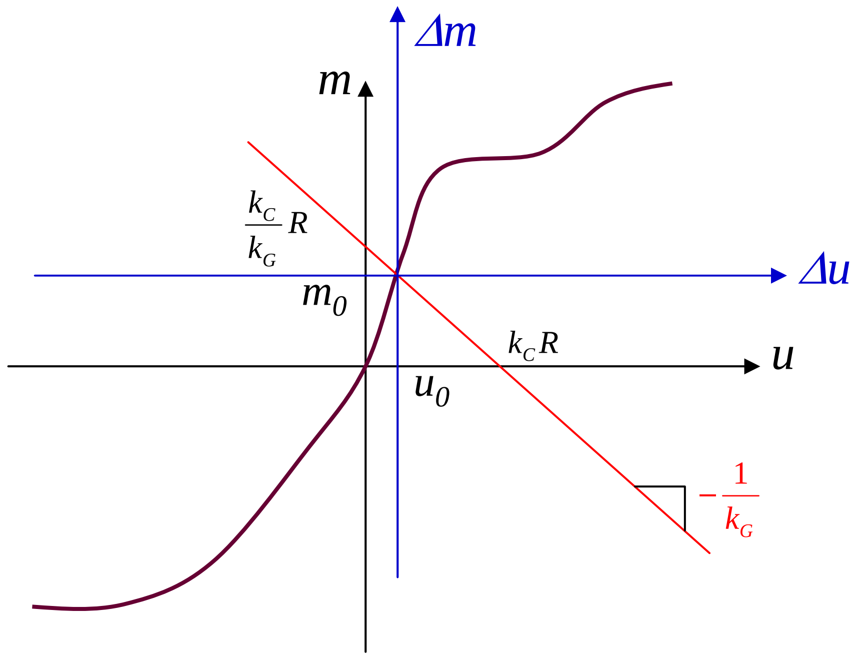
$$m(t) = m_0 + \Delta m(t)$$

$$w(t) = w_0 + \Delta w(t)$$

$$u_0 = k_C R - k_G m_0$$

System definition and problem statement

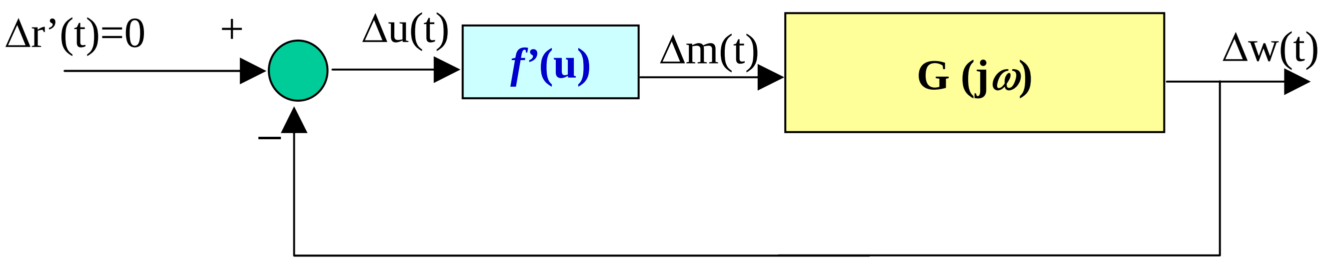
The non linear characteristics is translated so that the origin of the new reference Cartesian system is the point (u_0, m_0) in the original one



$$\begin{cases} m_0 = f(u_0) \\ u_0 = k_C R - k_G m_0 \end{cases}$$

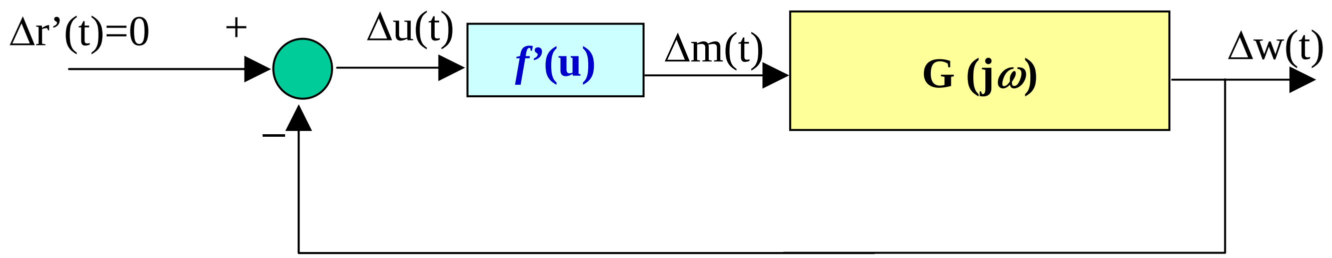
$$\Delta m_0 = f'(\Delta u)$$

$$f'(\Delta u) = f(u_0 + \Delta u) - m_0$$



System definition and problem statement

The aim of the system analysis is to define which is the steady-state behaviour of the system defined by the variations around the nominal working point

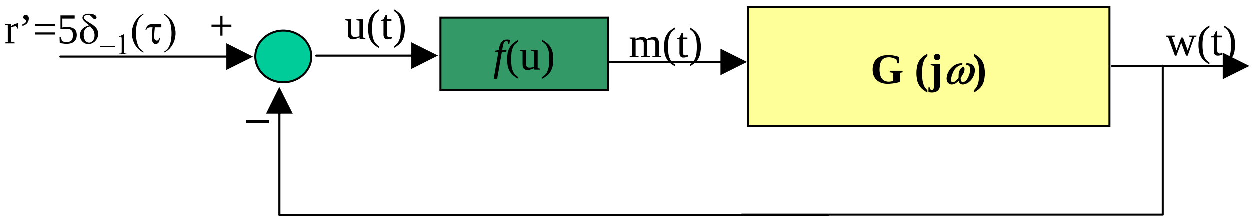


If $f'(u)$ is a passive sector function **absolute stability** tools allow for **sufficient conditions** for global asymptotic stability of the variation system, i.e, the steady state is characterised by constant values of the system variables

If the origin of the variation system is not stable does the variables diverge to infinity or some **periodic motion can appear**?

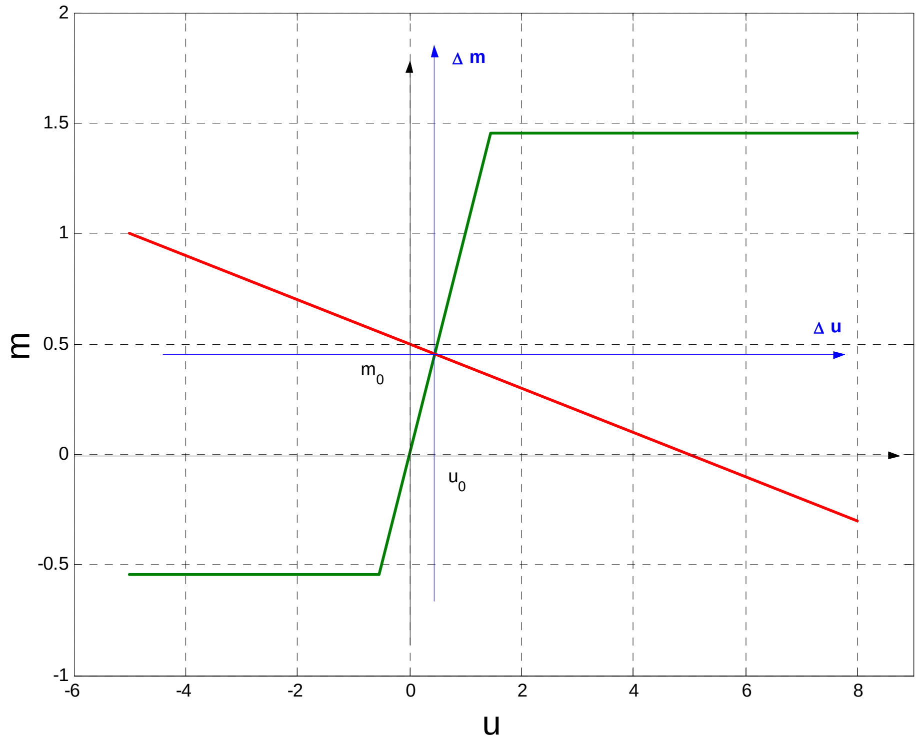
Limit cycles

Limit cycle: a periodic oscillation around a constant working point



$$G(s) = \frac{10}{(s+1)(s^2+1.4s+1)}$$

$$f(u) = sat\left(1, -\tfrac{6}{11}, \tfrac{16}{11}\right)$$



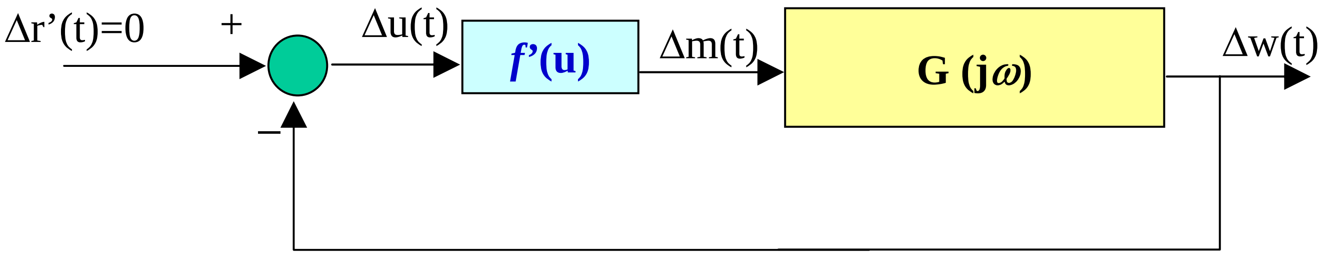
$$m = -\frac{1}{10}u + \frac{1}{2}$$

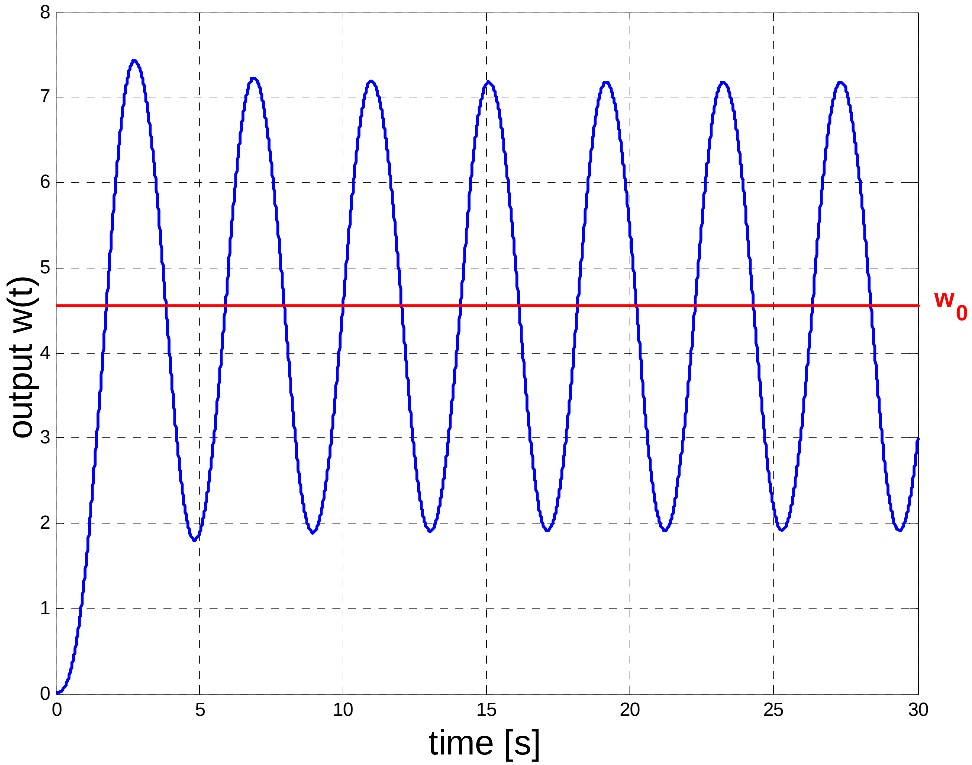
$$u_0 = \frac{5}{11}$$

$$m_0 = \frac{5}{11}$$

Limit cycles

Limit cycle: a periodic oscillation around a constant working point





$$u_0 = \frac{5}{11}$$

$$m_0 = \frac{5}{11}$$

$$w_0 = \frac{50}{11} = 4.545$$

Limit cycles

Limit cycle can be a drawback in control systems:

- ✓ *Instability of the equilibrium point*
- ✓ *Wear and failure in mechanical systems*
- ✓ *Loss of accuracy in regulation*

Parameters of the limit cicle can be used to discriminate between acceptable and dangerous oscillations

- ➢ oscillation frequency
- ➢ oscillation magnitude

Electronic oscillators can be based on limit cycles

Describing Function - *Assumptions*

The describing Function approach to the analysis of steady-state oscillations in non linear systems is an approximate tool to estimate the limit cycle parameters.

It is based on the following assumptions

- ✓ There is only one single nonlinear component
- ✓ The nonlinear component is not dynamical and time invariant
- ✓ The linear component has low-pass filter properties
- ✓ The nonlinear characteristic is symmetric with respect to the origin

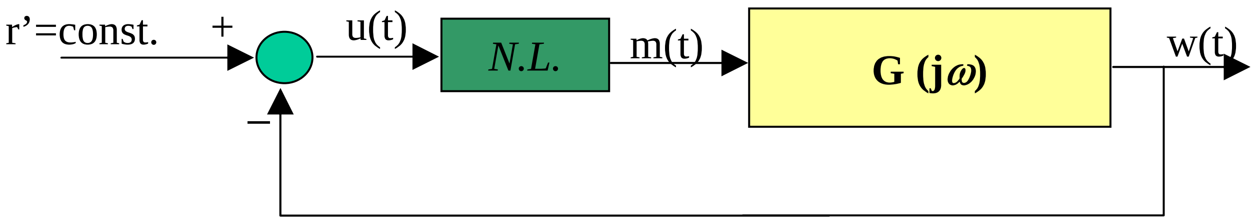
Describing Function - *Assumptions*

There is only one single nonlinear component

The system can be represented by a lumped parameters system with two main blocks:

•The linear part

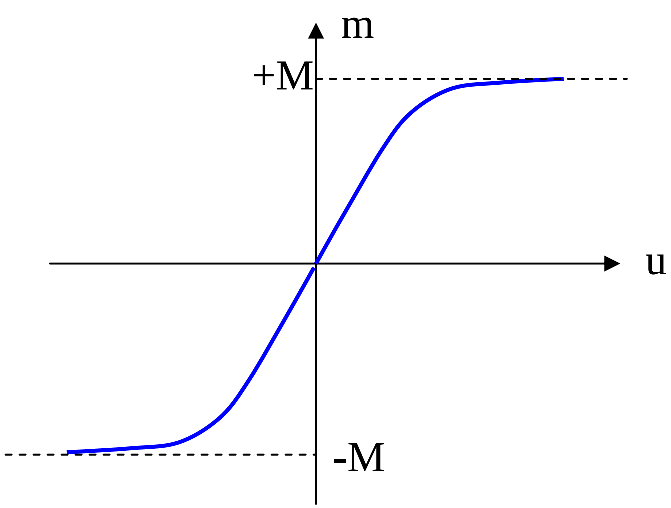
•The nonlinear part



Describing Function - *Assumptions*

The nonlinear component is not dynamical and time invariant

The system is autonomous.
All the system dynamics is concentrated in the linear part so that classical analysis tools such as Nyquist and Bode plots can be applied.



$$m = M\frac{u}{|u| + \varepsilon}$$

Describing Function - *Assumptions*

The linear component has low-pass filter properties

This is the main assumption that allows for neglecting the higher frequency harmonics that can appear when a nonlinear system is driven by a harmonic signal

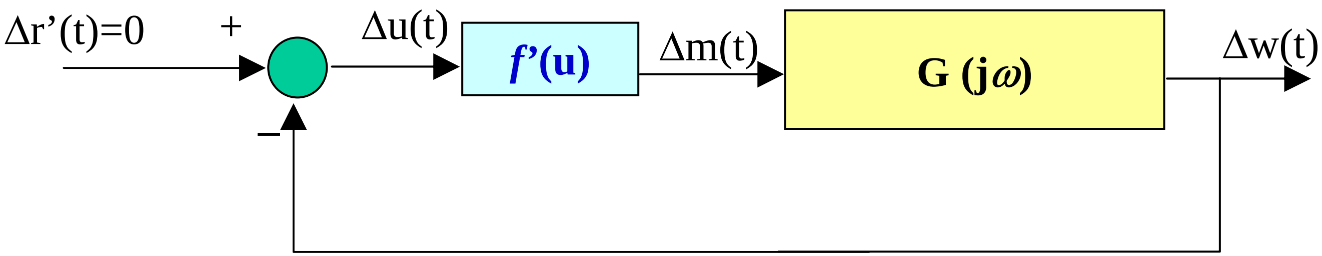
$$\left|G\left(j\overline{\omega}\right)\right| >> \left|G\left(jn\overline{\omega}\right)\right| \quad n = 2, 3, \ldots$$

The more the low-pass filter assumption is verified the more the estimation error affecting the limit cycle parameters is small

Describing Function - *Assumptions*

The nonlinear characteristic is symmetric with respect to the origin

This guarantees that the static term in the Fourier expansion of the output of the nonlinearity, subjected to an harmonic signal, can be neglected



Such an assumption is usually taken for the sake of simplicity, and it can be relaxed

Fourier expansion - *Recall*

Consider a periodic function

$$y(t) = f(t), \quad y(t) = y(t-T), \quad T \text{ is a real constant}$$

$$\Downarrow$$

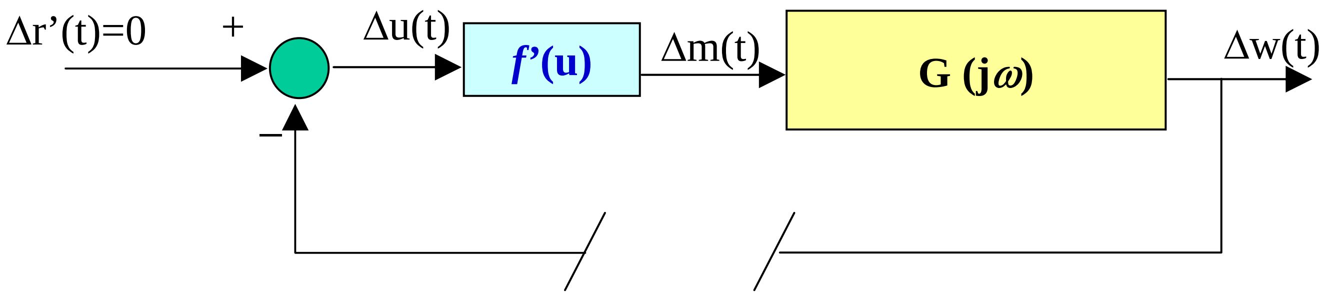
$$y(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \sin\left(k \tfrac{2\pi}{T} t\right) + b_k \cos\left(k \tfrac{2\pi}{T} t\right) \right)$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos\left(k \tfrac{2\pi}{T} t\right) dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \cos\left(k \tfrac{2\pi}{T} t\right) d\left(\tfrac{2\pi}{T} t\right)$$

$$k = 0,1,2,\ldots \quad b_0 = 0$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin\left(k \tfrac{2\pi}{T} t\right) dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \sin\left(k \tfrac{2\pi}{T} t\right) d\left(\tfrac{2\pi}{T} t\right)$$

Describing Function – *Harmonic balance*



$$\Delta u(t) = U \sin\left(\bar{\omega} t\right)$$

$$\Downarrow$$

$$\Delta m(t) = \sum_{k=1}^{\infty} \left(a_k \cos\left(k\bar{\omega} t\right) + b_k \sin\left(k\bar{\omega} t\right)\right)$$

$$\Downarrow$$

$$\Delta w(t) = \sum_{k=1}^{\infty} G_k \left(a_k \cos\left(k\bar{\omega} t + \varphi_k\right) + b_k \sin\left(k\bar{\omega} t + \varphi_k\right)\right)$$

$$G_k = \left|G\left(jk\bar{\omega}\right)\right| \qquad \varphi_k = \angle G\left(jk\bar{\omega}\right)$$

Describing Function – *Harmonic balance*

Consider the polar representation of a complex number associated with the exponential form of harmonic signals

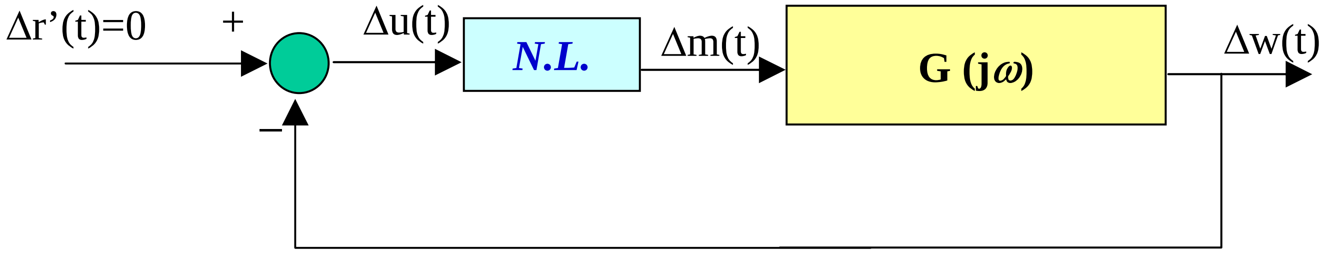
$$\Delta u(t) = U e^{j\overline{\omega}t}$$

$$m(t) = \sum_{k=1}^{\infty} M_k e^{j\vartheta_k} e^{jk\overline{\omega}t} \qquad M_k = \sqrt{a_k^2 + b_k^2} \qquad \vartheta_k = \arctan\frac{a_k}{b_k}$$

Taking into account the low-pass property of the linear part of the system

$$\Delta w(t) = \sum_{k=1}^{\infty} G_k e^{j\varphi_k} M_k e^{j\vartheta_k} e^{jk\overline{\omega}t} \simeq G_1 e^{j\varphi_1} M_1 e^{j\vartheta_1} e^{j\overline{\omega}t}$$

Describing Function – *Harmonic balance*



A permanent oscillation in the loop appears if Δu(t)=-Δw(t)

$$Ue^{j\overline{\omega}t} = -G_1 e^{j\varphi_1} M_1 e^{j\vartheta_1} e^{j\overline{\omega}t} \quad \Longrightarrow \quad 1 + G_1 e^{j\varphi_1} \frac{M_1}{U} e^{j\vartheta_1} = 0$$

Harmonic balance equation

$$1 + G(j\omega) N(U,\omega) = 0$$

$$N(U,\omega) = \frac{1}{U}\left(b_1 + ja_1\right)$$

is the **D**escribing **F**unction of the nonlinear term

Describing Function – *Harmonic balance*

The armonic balance equation is a **necessary condition** for the existence of limit cycles in the nonlinear system

The **approximate** analysis gives good estimates if the low-pass filter hypothesis is strongly verified. It is a good tools for engineers

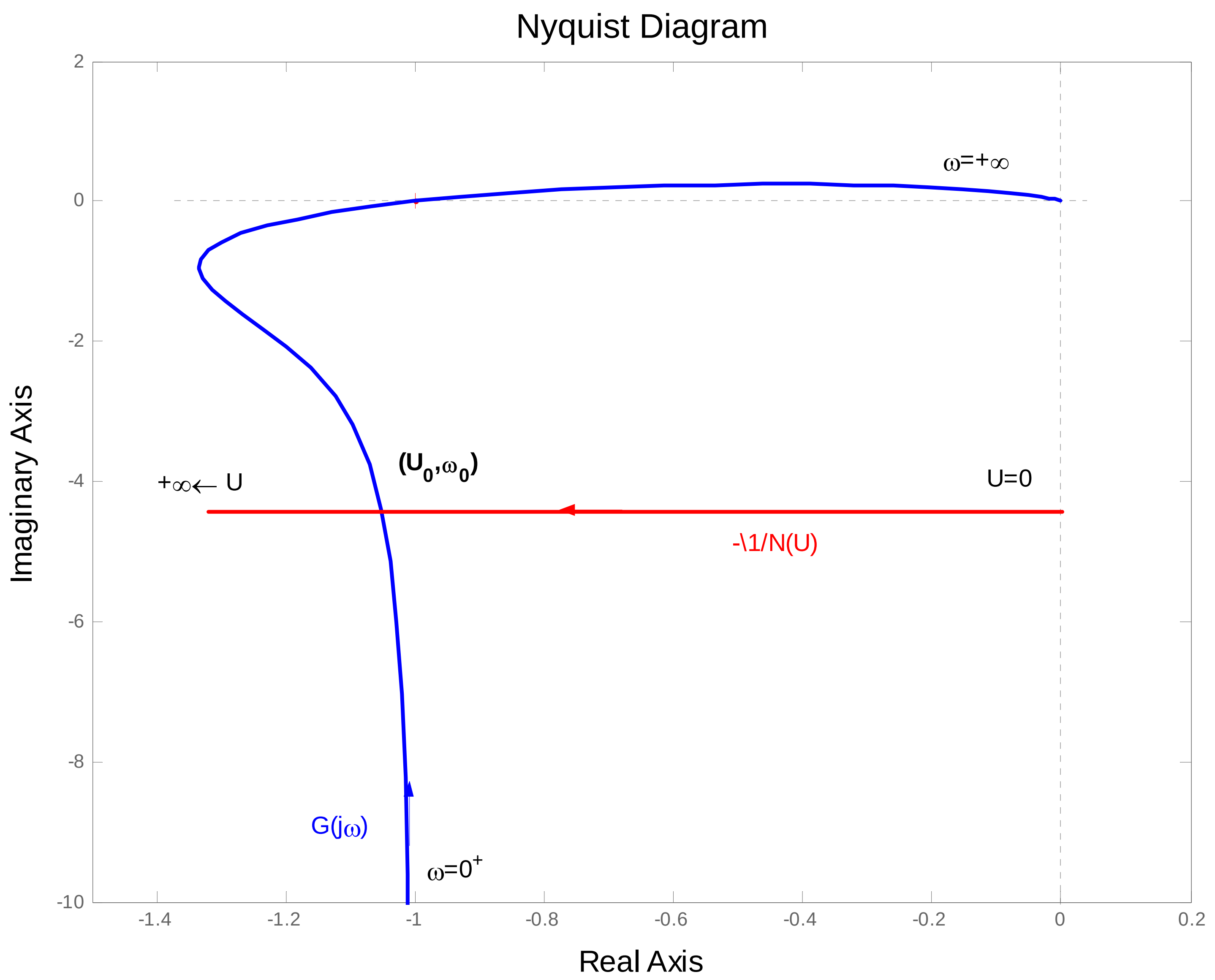
The harmonic balance equation is similar to the characteristic polynomial function, i.e. it leads to the Nyquist condition for closed-loop stability

The Describing Function is a linear approximation of the static nonlinearity limited to the first harmonic

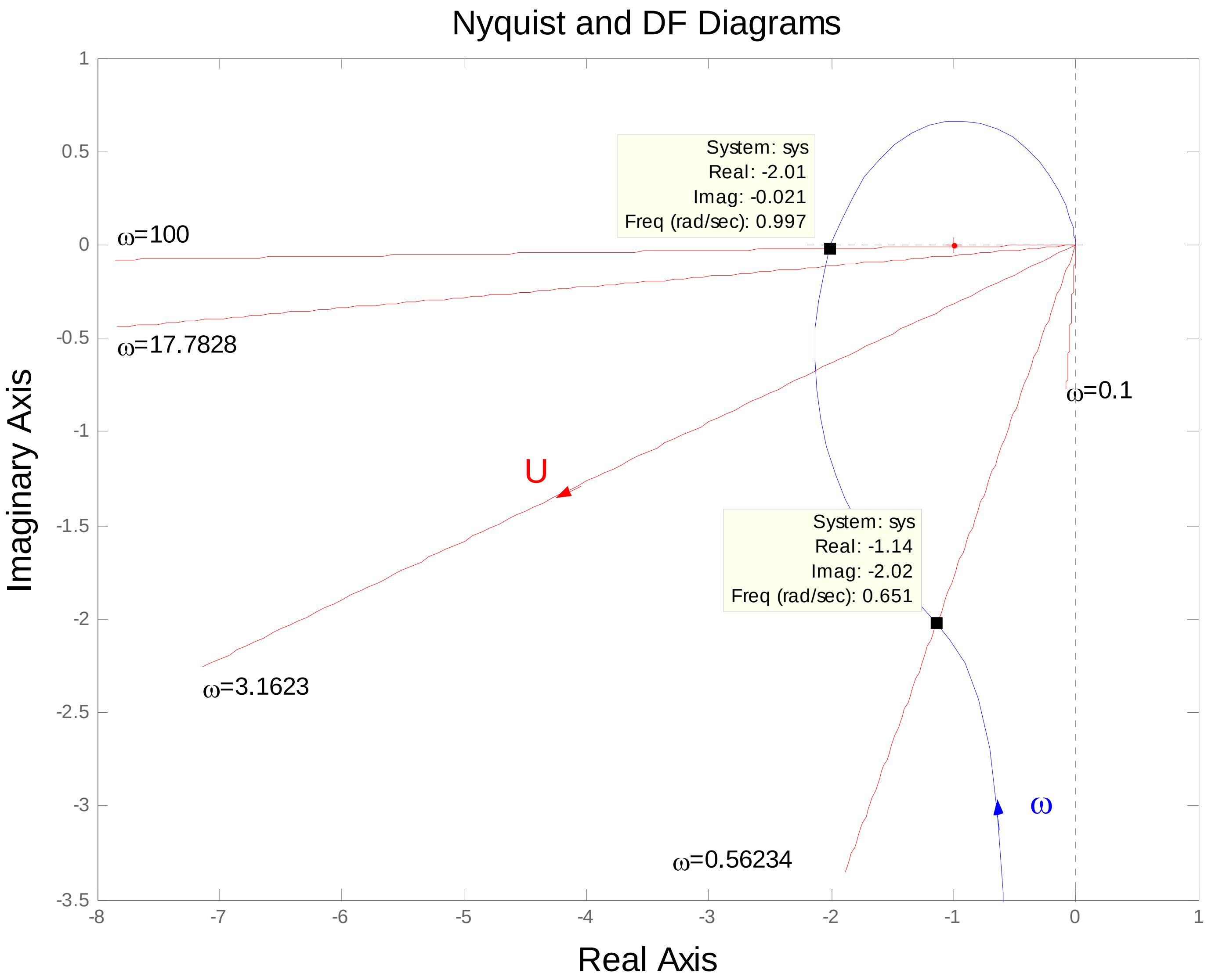
In most cases the Describing Function is not a function of the frequency and this simplifies the verification of the harmonic balance equation by means of the Nyquist plot of the transfer function

$$G(j\omega) = -\frac{1}{N(U,\omega)}$$

Describing Function – *Harmonic balance*

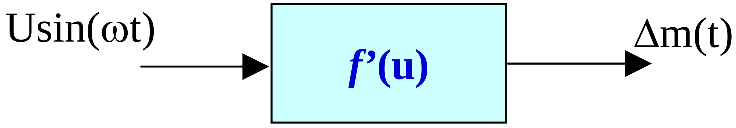


Describing Function – *Harmonic balance*



Describing Function – *Computation*

The DF computation can be performed by means of its definition



$$N(U,\omega)=\frac{1}{U}\left(b_1+ja_1\right)$$

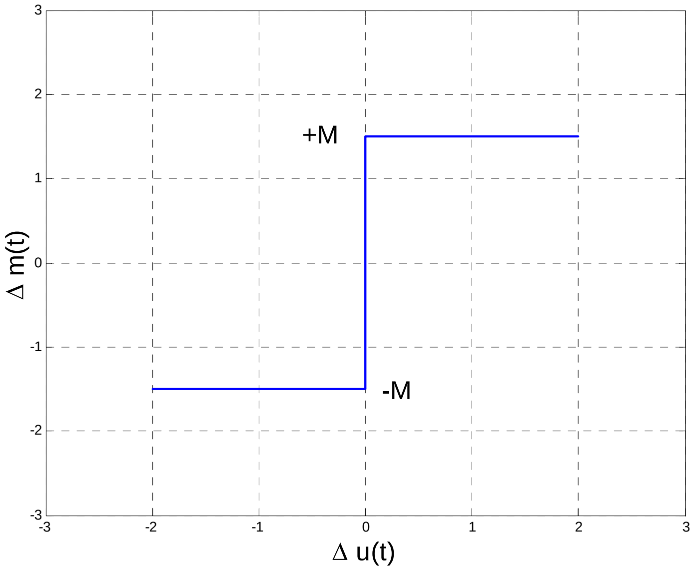
$$a_1=\frac{2}{T}\int_{-T/2}^{T/2}\Delta m(t)\cos(\omega t)\,dt$$

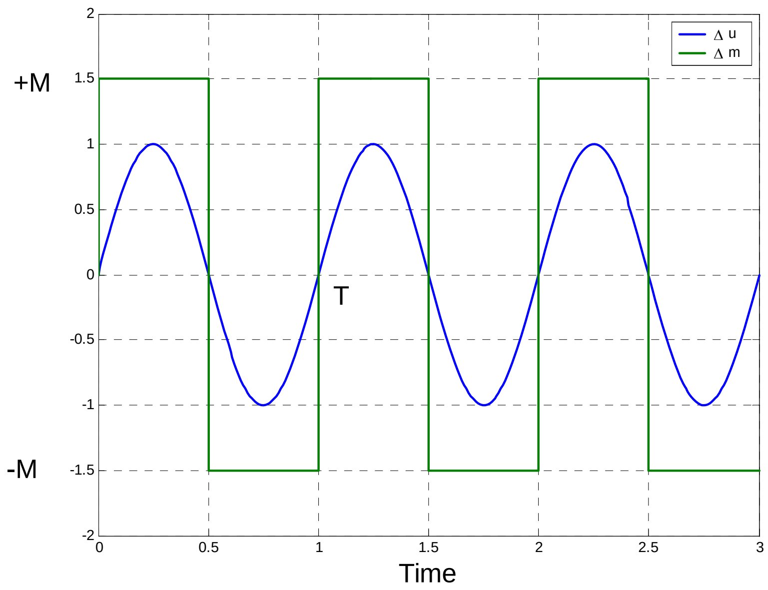
$$b_1=\frac{2}{T}\int_{-T/2}^{T/2}\Delta m(t)\sin(\omega t)\,dt$$

The evaluation of coefficients a_1 and b_1 can be performed by means of both analytical calculation and numerical integration, depending on the type of nonlinearity involved

Describing Function – *Computation*

Ideal relay





$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} \Delta m(t)\cos(\omega t)dt = 0$$

Because of the odd symmetry of the Δm(t) signal

$$b_1 = \frac{2}{T}\int_{-T/2}^{T/2} \Delta m(t)\sin(\omega t)dt = \frac{4}{\pi}\int_{0}^{\pi/2} M\sin(\vartheta)d\vartheta = \frac{4M}{\pi}$$

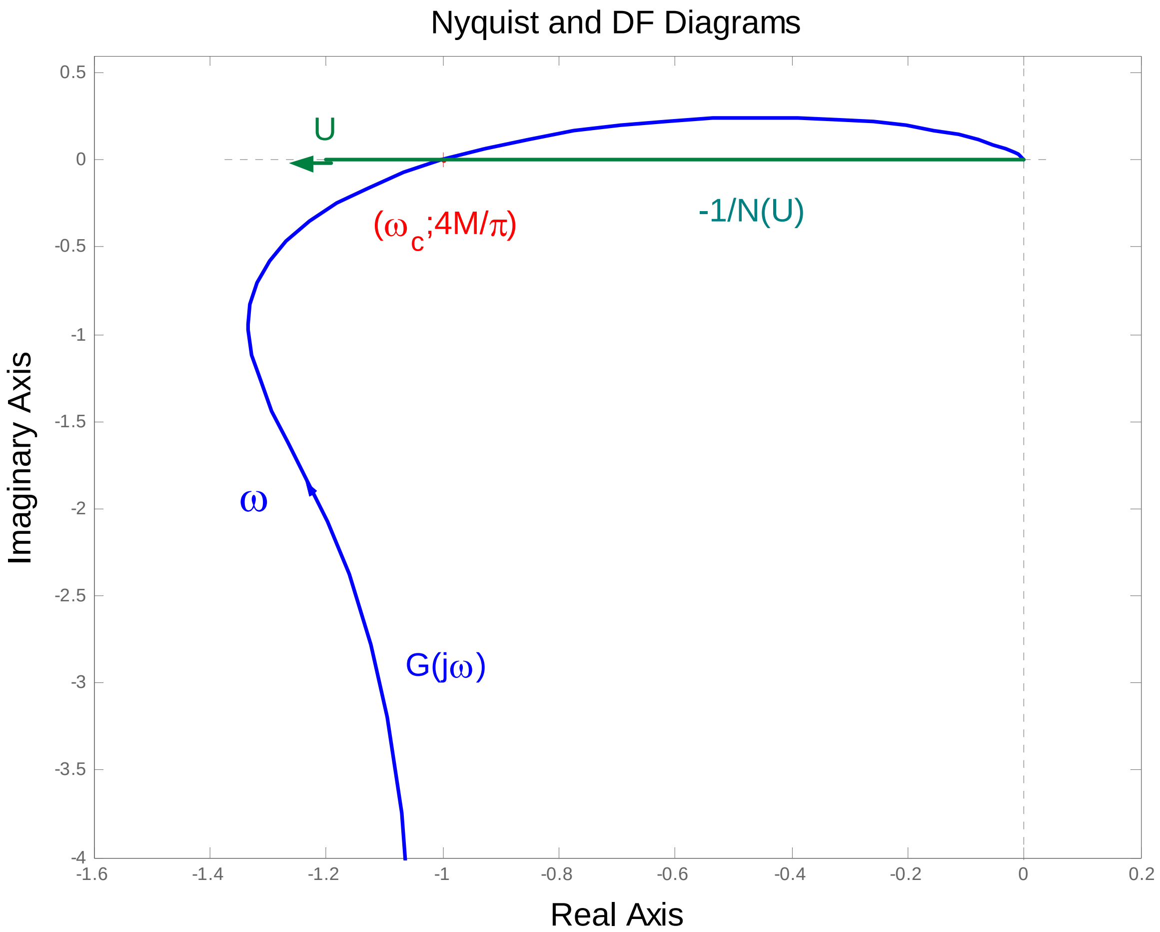
$$N(U,\omega) = \frac{4M}{U\pi}$$

Describing Function – *Computation*

Ideal relay

The negative reciprocal of the DF is the negative real axis in backward direction

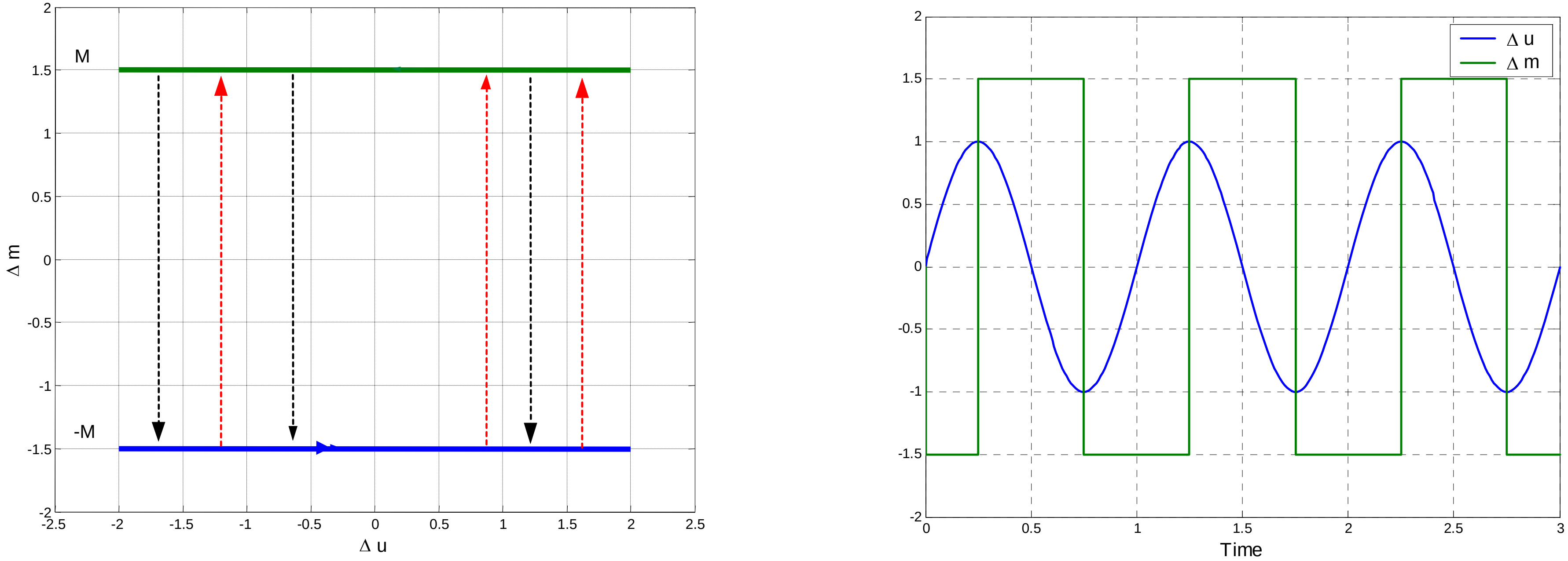
A limit cycle can exist if the relative degree of G(jω) is greater than two



The oscillation frequency is the critical frequency ω_c of the linear system and the oscillation magnitude is proportional to the relay gain M

Describing Function – *Computation*

Pure hysteresis



$$b_1 = \frac{2}{T}\int_{-T/2}^{T/2} \Delta m(t)\cos(\omega t)dt = 0$$

Because of the even symmetry of the Δm(t) signal

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} \Delta m(t)\cos(\omega t)dt = -\frac{2}{\pi}\int_{0}^{\pi/2} M\cos(\vartheta)d\vartheta = -\frac{4M}{\pi}$$

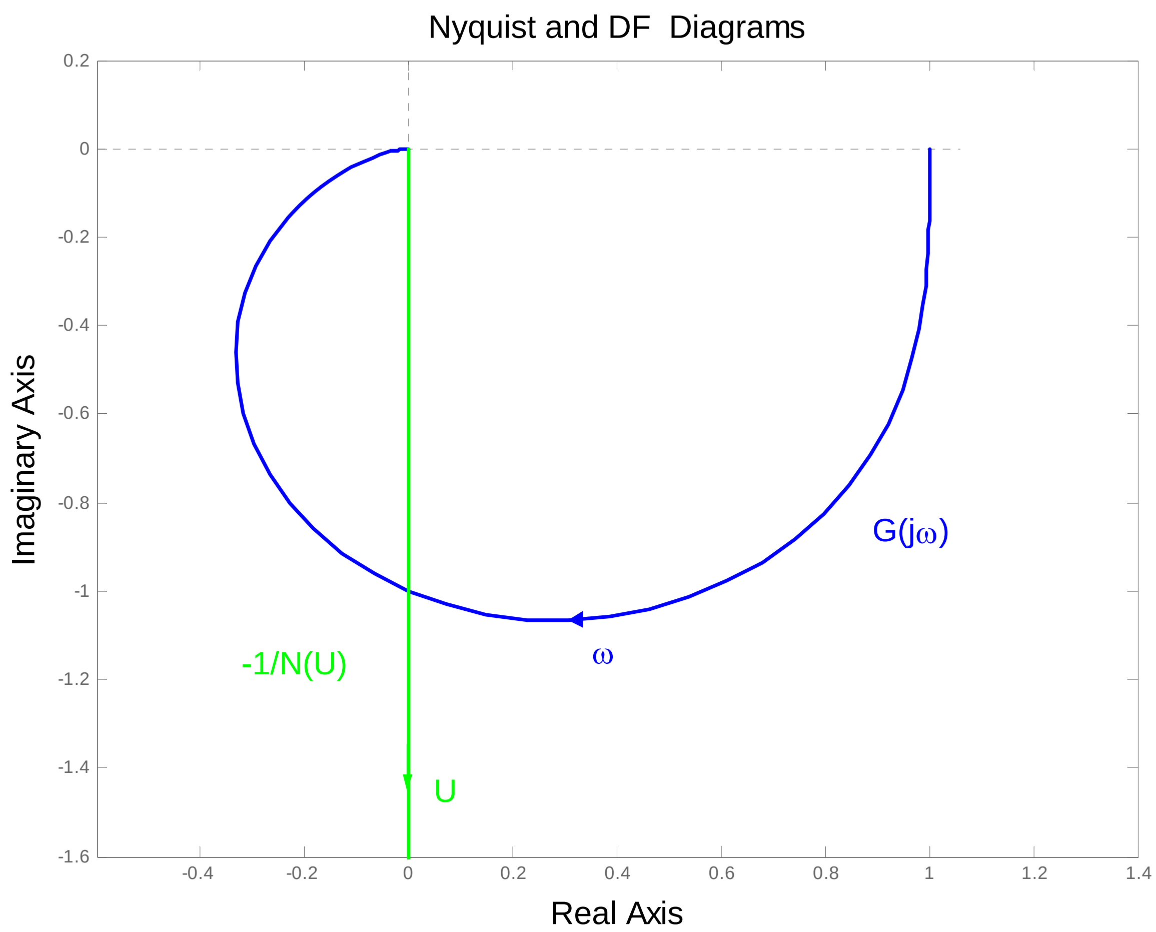
$$N(U,\omega) = -j\frac{4M}{U\pi}$$

Describing Function – *Computation*

Pure hysteresis

The negative reciprocal of the DF is the negative imaginary axis in backward direction

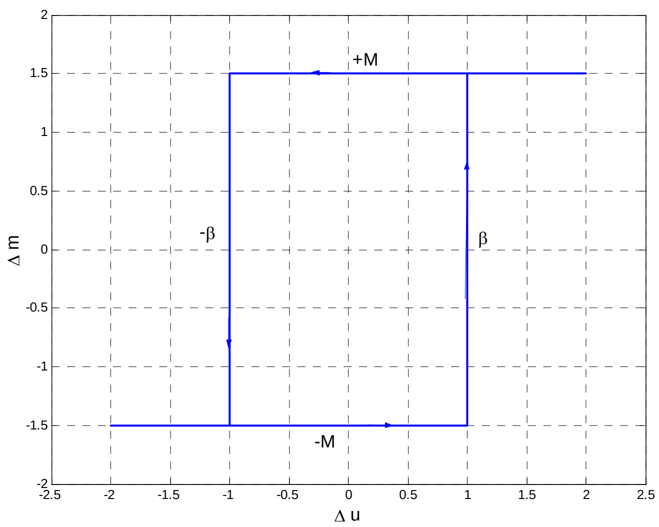
A limit cycle can exist if the relative degree of G(jω) is greater than one and G(jω) is a type-0 system

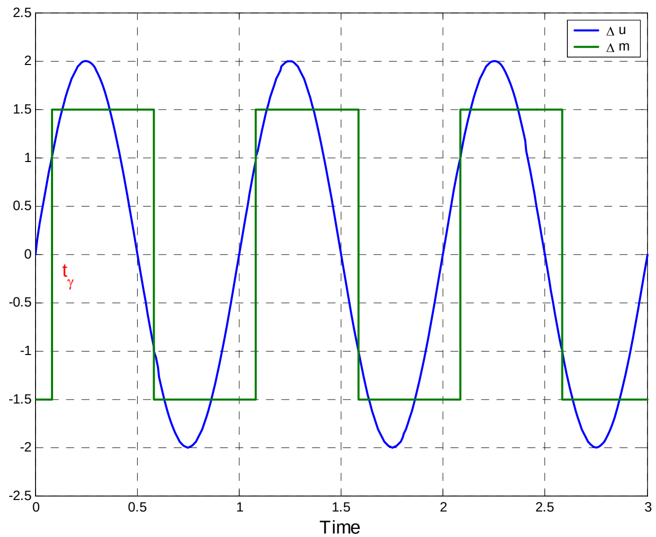


The oscillation frequency is lower than the critical frequency ω_c of the linear system and the oscillation's magnitude is proportional to the relay gain M and to the modulus of the transfer function at phase $-\pi/4$

Describing Function – *Computation*

Hysteretic relay





If U≤β the hysteretic relay behaves as pure hysteresis

$$b_1 = -M\frac{8}{T}\int_0^{t_\gamma}\sin(\omega t)dt + M\frac{8}{T}\int_{t_\gamma}^{T/4}\sin(\omega t)dt = \frac{4M}{\pi}\sqrt{1-\left(\frac{\beta}{U}\right)^2}$$

$$a_1 = -M\frac{8}{T}\int_0^{t_\gamma}\cos(\omega t)dt + M\frac{8}{T}\int_{t_\gamma}^{T/4}\cos(\omega t)dt =$$

$$N(U,\omega) = \begin{cases} -j\dfrac{4M}{U\pi} & U \le \beta \\ \dfrac{4M}{\pi U}\sqrt{1-\left(\dfrac{\beta}{U}\right)^2} - j\dfrac{4M\beta}{\pi U^2} & U > \beta \end{cases}$$

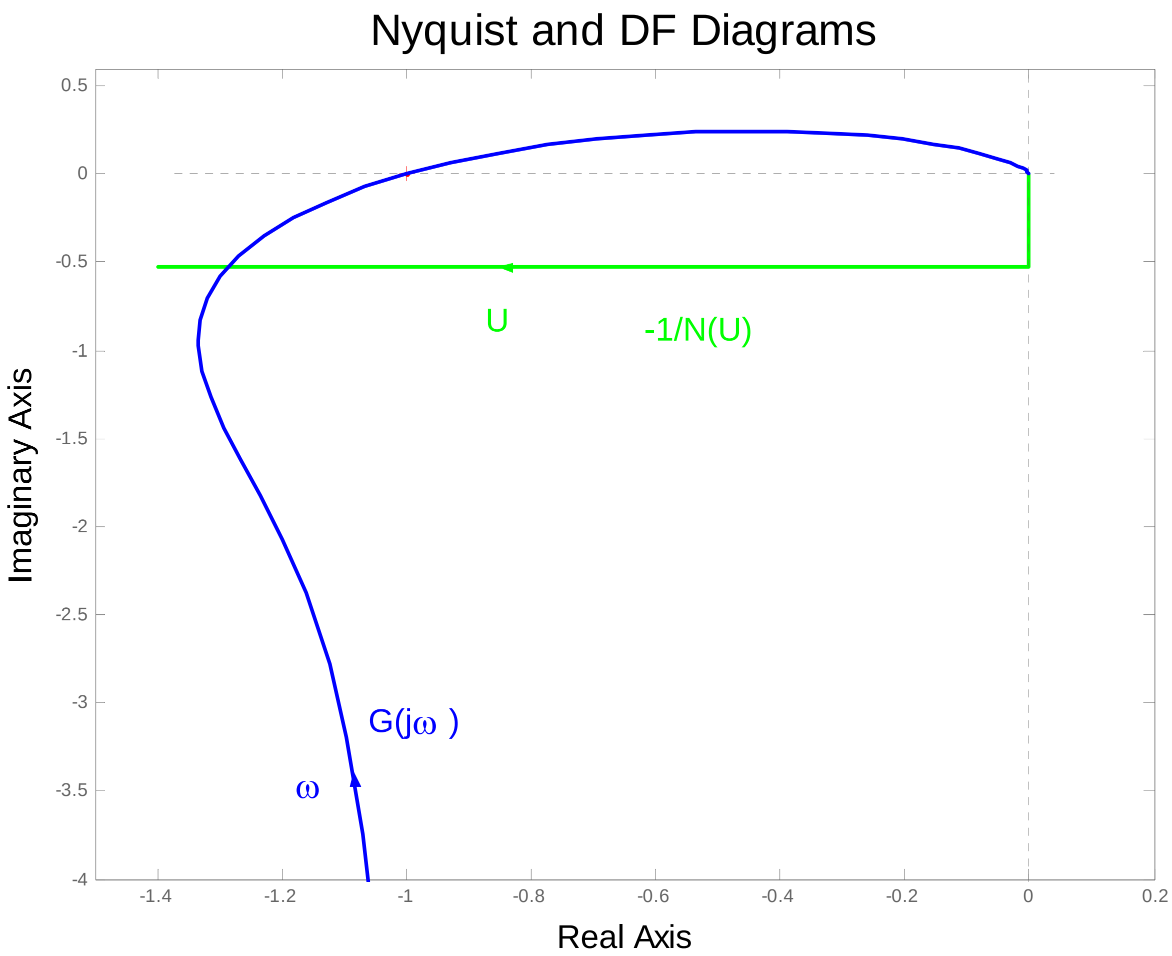
The imaginary part of N(U) is proportional to the hysteresis area

Describing Function – *Computation*

Hysteretic relay

The negative reciprocal of the DF is the parallel to the negative real axis and with constant negative imaginary part

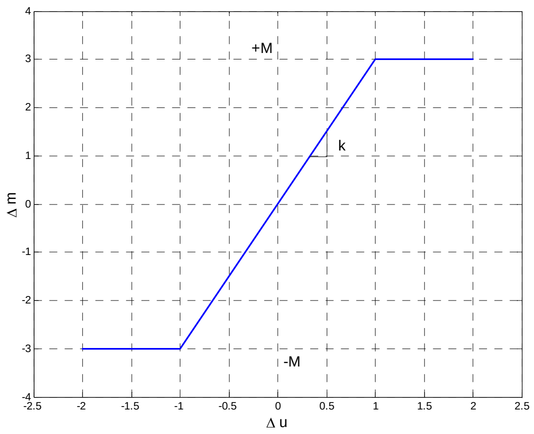
If β is larger than U the relay could not behave as a pure hysteresis

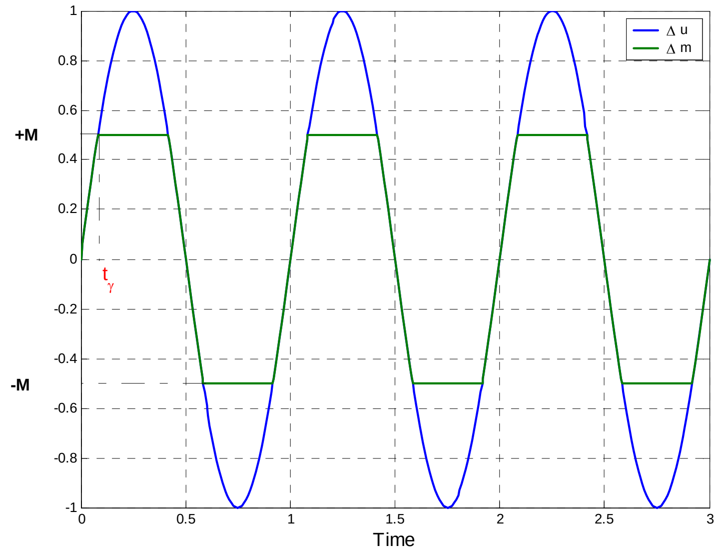


The oscillation frequency is lower than the critical frequency ω_c of the linear system and the oscillation's magnitude is proportional to the relay gain M

Describing Function – *Computation*

Saturation





If U≤M the saturation behaves as pure gain

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} \Delta m(t)\cos(\omega t)\,dt = 0$$

Because of the odd symmetry of the Δm(t) signal

$$b_1 = \frac{8}{T}\int_{0}^{t_\gamma} kU\sin^2(\omega t)\,dt + M\,\frac{8}{T}\int_{t_\gamma}^{T/4}\sin(\omega t)\,dt = \frac{2kU}{\pi}\left[\arcsin\left(\frac{M}{kU}\right) + \left(\frac{M}{kU}\right)\sqrt{1-\left(\frac{M}{kU}\right)^2}\right]$$

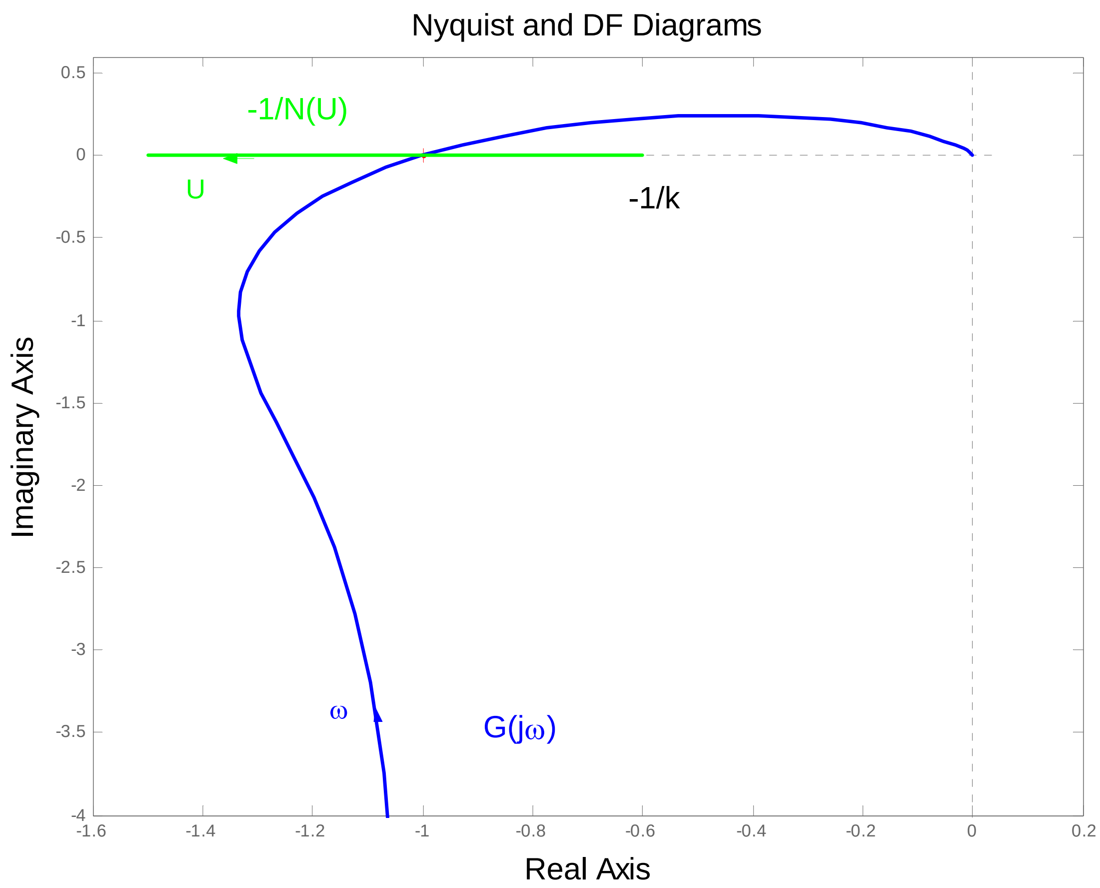
$$N(U) = \begin{cases} k & U \le \dfrac{M}{k} \\ \dfrac{2k}{\pi}\left[\arcsin\left(\dfrac{M}{kU}\right) + \left(\dfrac{M}{kU}\right)\sqrt{1-\left(\dfrac{M}{kU}\right)^2}\right] & U > \dfrac{M}{k} \end{cases}$$

Describing Function – *Computation*

Saturation

The negative reciprocal of the DF is part of the negative real axis in backward direction

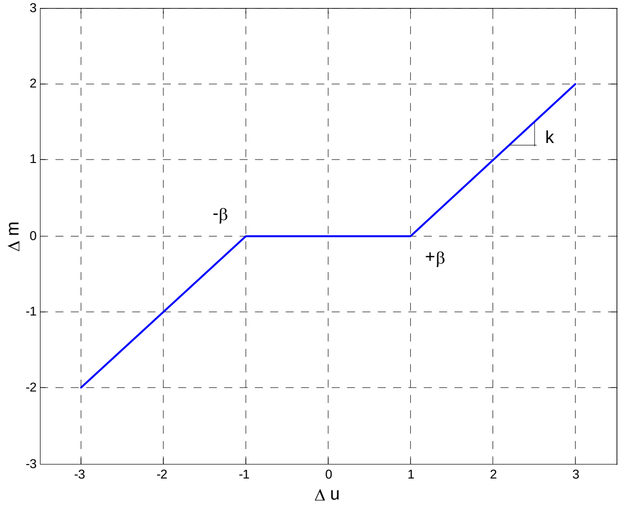
A limit cycle can exist if the relative degree of G(jω) is greater than two and gain *k* is sufficiently high

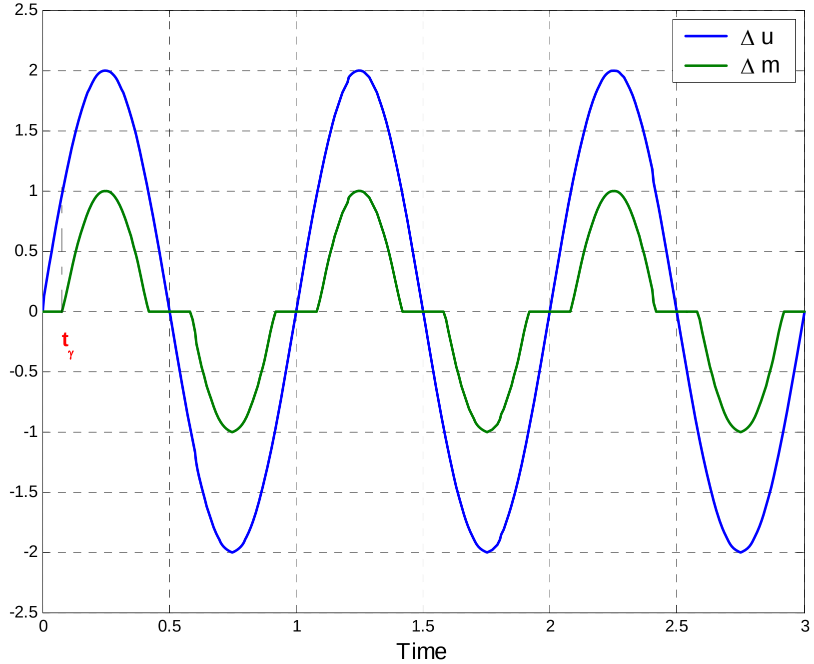


The oscillation frequency is the critical frequency ω_c of the linear system and the oscillation's magnitude depends on the saturation parameters M and k

Describing Function – *Computation*

Dead zone





If U≤β the dead zone has no output

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} \Delta m(t)\cos(\omega t)dt = 0$$

Because of the odd symmetry of the Δm(t) signal

$$b_1 = \frac{8}{T}\int_{t_\gamma}^{T/4} k\left(U\sin(\omega t) - \beta\right)\sin(\omega t)dt = \frac{2kU}{\pi}\left[\frac{\pi}{2} - \arcsin\left(\frac{\beta}{U}\right) - \left(\frac{\beta}{U}\right)\sqrt{1-\left(\frac{\beta}{U}\right)^2}\right]$$

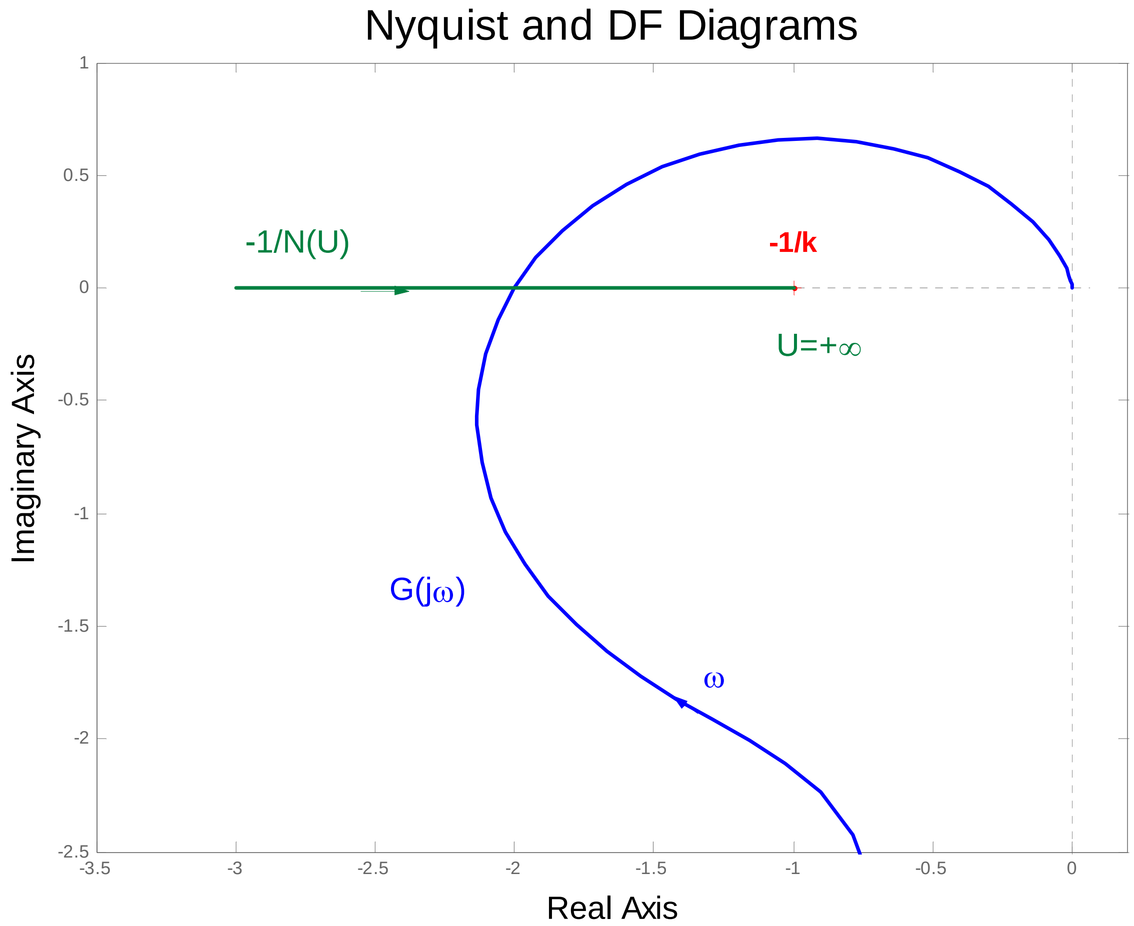
$$\beta = U\sin\left(\omega t_\gamma\right)$$

$$N(U) = \begin{cases} 0 & U \le \beta \\ k - \dfrac{2k}{\pi}\left[\arcsin\left(\dfrac{\beta}{U}\right) + \left(\dfrac{\beta}{U}\right)\sqrt{1-\left(\dfrac{\beta}{U}\right)^2}\right] & U > \beta \end{cases}$$

Describing Function – *Computation*

Dead zone

The negative reciprocal of the DF is part of the negative real axis in forward direction



A limit cycle can exist if the relative degree of G(jω) is greater than two and gain *k* is sufficiently high

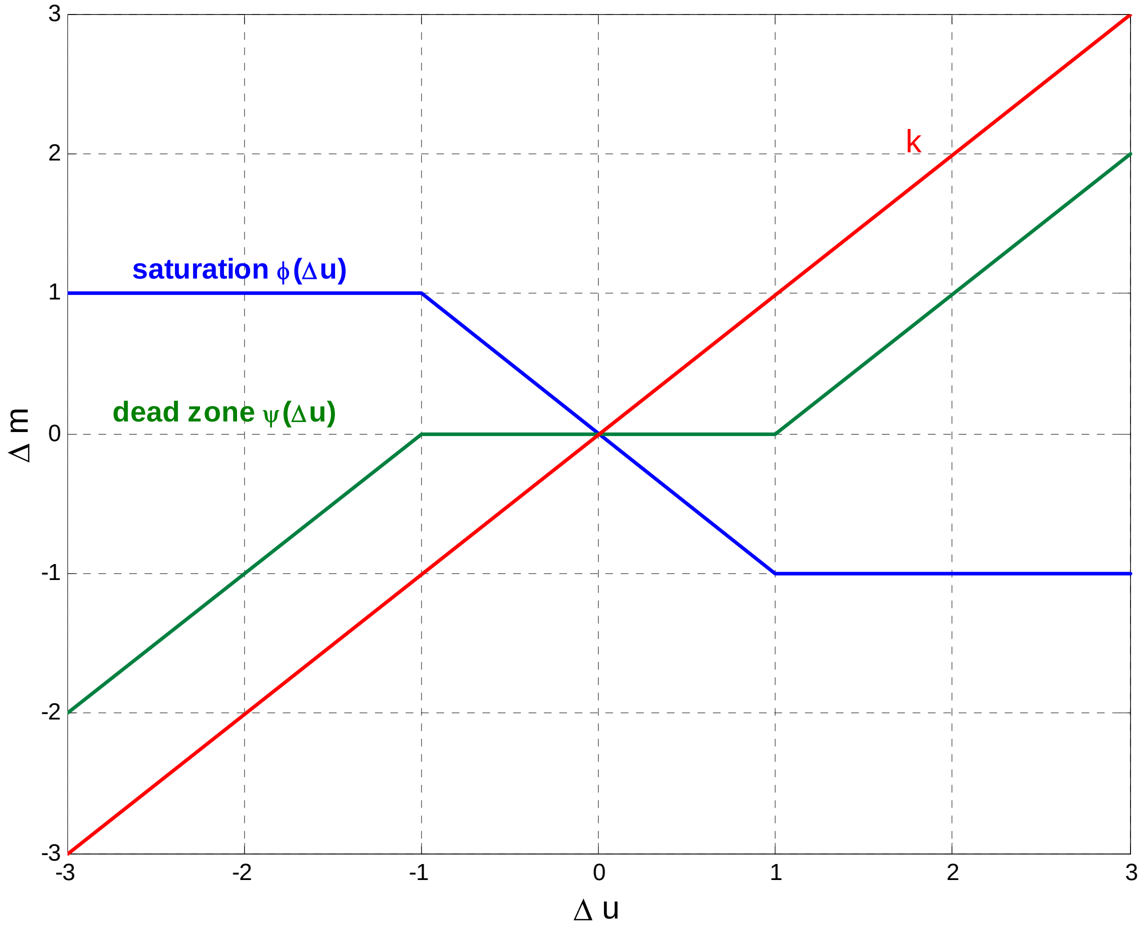
The oscillation frequency is the critical frequency ω_c of the linear system and the oscillation magnitude depends on the dead zone parameters β and *k*

Describing Function – *Computation*

Dead zone

The nonlinear characteristics of the Dead Zone can be computed by subtracting the Saturation characteristics from a linear one

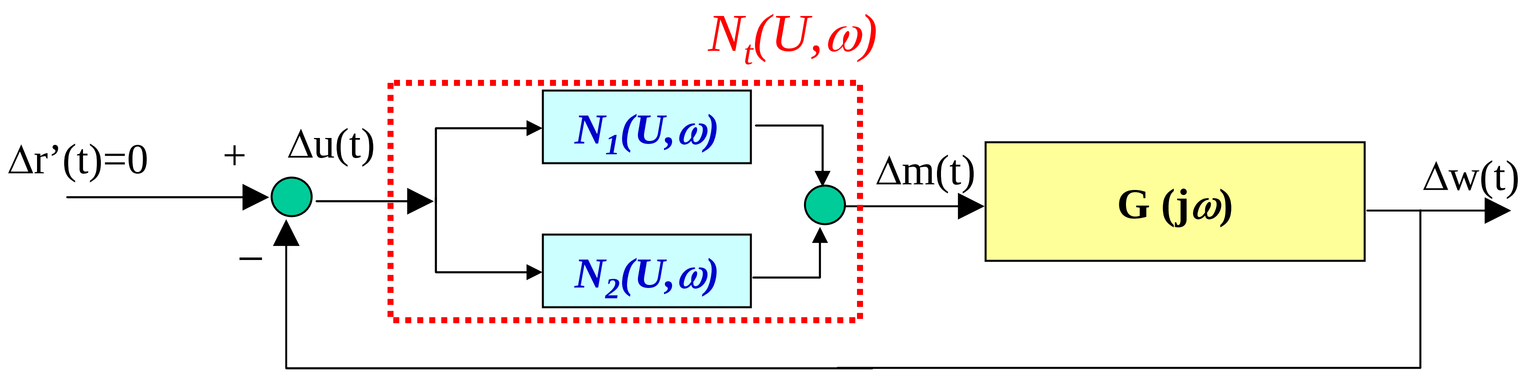
$$\Psi(\Delta u) = k - \Phi(\Delta u)$$



$$N_\Psi(U) = k - N_\Phi(U)$$

Describing Function – *Computation*

The Describing function of a nonlinear characteristics can be computed as the combination of the Describing Functions of the elementary constituting nonlinear characteristics



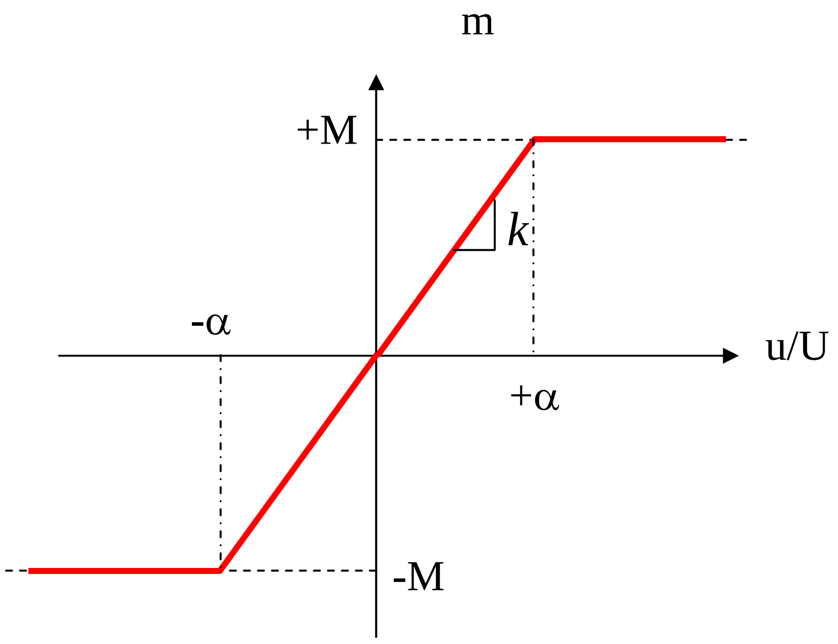
$$N_t(U) = N_1(U) + N_2(U)$$

Describing Function – *Computation*

A number of Describing Function can be computed by particularisation of the function

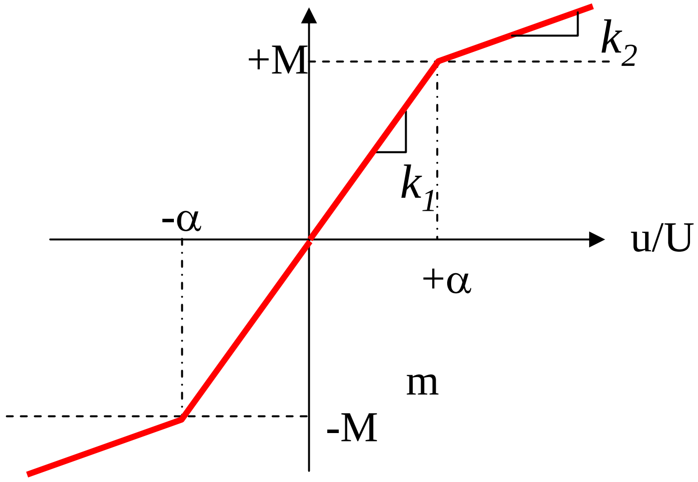
$$\Phi(\alpha) = \frac{2}{\pi}\left[\arcsin(\alpha) + \alpha\sqrt{1-\alpha^2}\right]$$

in which the parameter α defines a peculiar point of the nonlinear characteristics

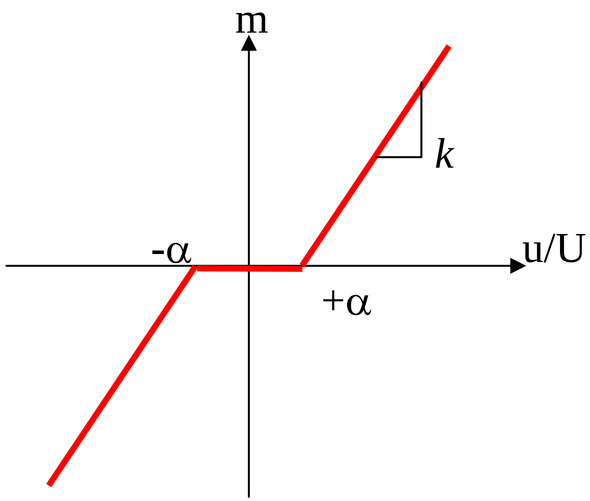


$$N(\alpha) = \begin{cases} k & \alpha \geq 1 \\ k \cdot \Phi(\alpha) & \alpha < 1 \end{cases}$$

Describing Function – *Computation*

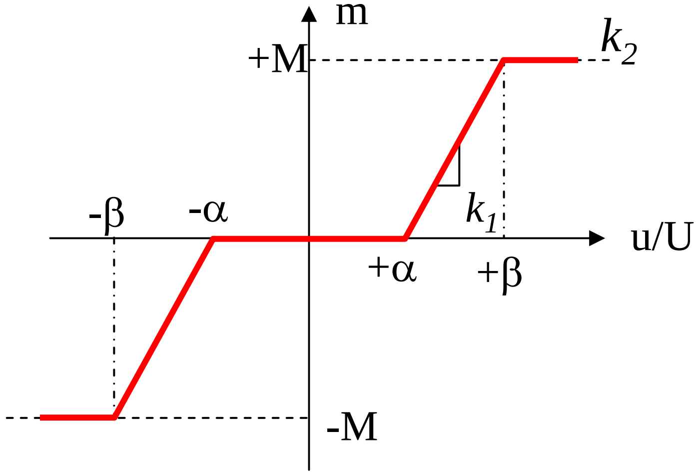


$$N(\alpha)=\begin{cases} k_1 & \alpha \geq 1 \\ k_2+(k_1-k_2)\cdot\Phi(\alpha) & \alpha < 1 \end{cases}$$

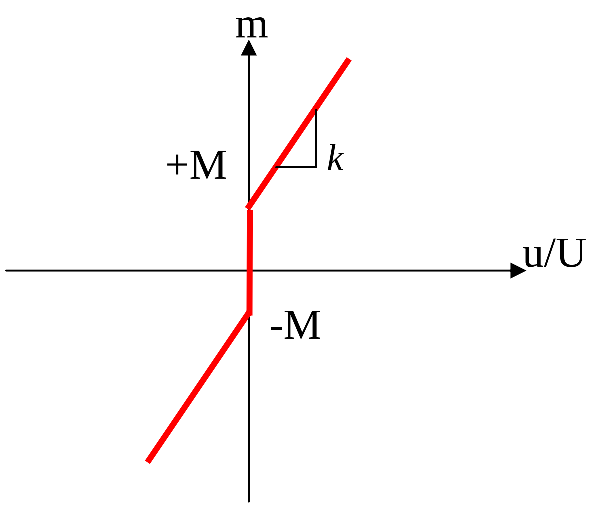


$$N(\alpha)=\begin{cases} 0 & \alpha \geq 1 \\ k(1-\Phi(\alpha)) & \alpha < 1 \end{cases}$$

Describing Function – *Computation*

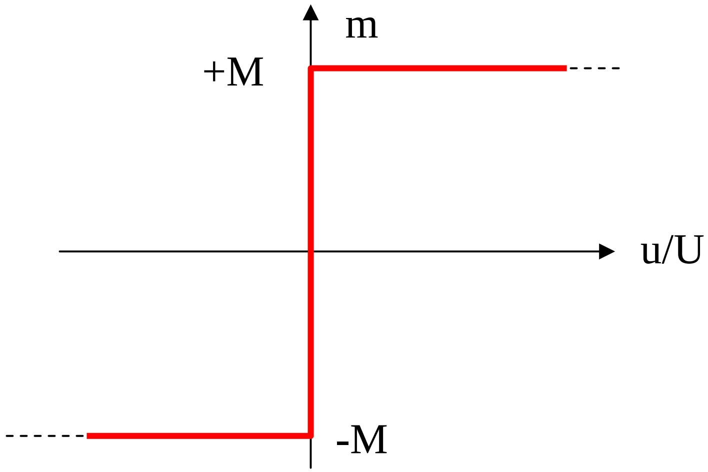


$$N(\alpha)=\begin{cases}0 & \alpha\geq 1\\ k\cdot\left(1-\Phi(\alpha)\right) & \alpha<1\leq\beta\\ k\cdot\left(\Phi(\beta)-\Phi(\alpha)\right) & \beta<1\end{cases}$$

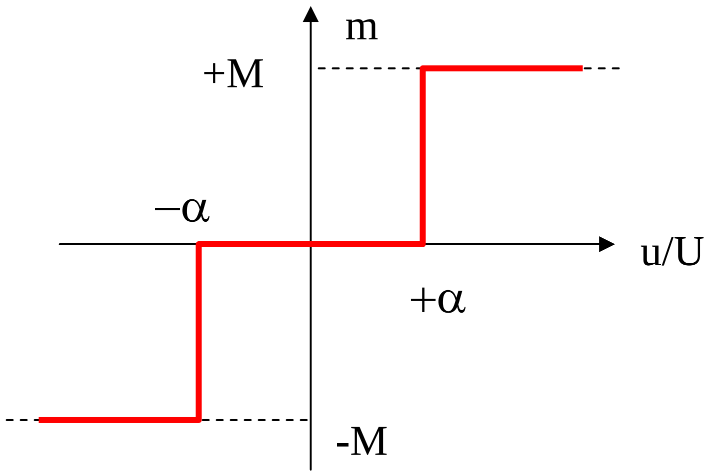


$$N(\alpha)=k+\frac{4M}{\pi}$$

Describing Function – *Computation*

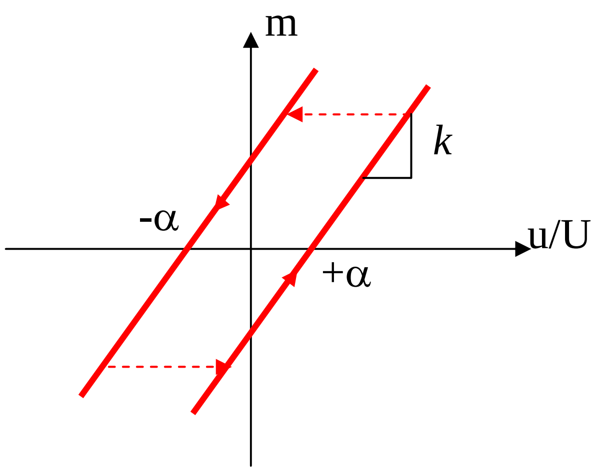


$$N(\alpha) = \frac{4M}{\pi}$$

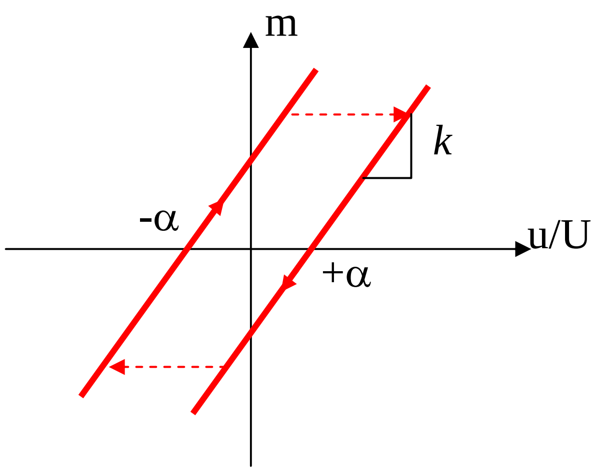


$$N(\alpha) = \begin{cases} 0 & \alpha \geq 1 \\ \dfrac{4M}{\pi}\sqrt{1-\alpha^2} & \alpha < 1 \end{cases}$$

Describing Function – *Computation*

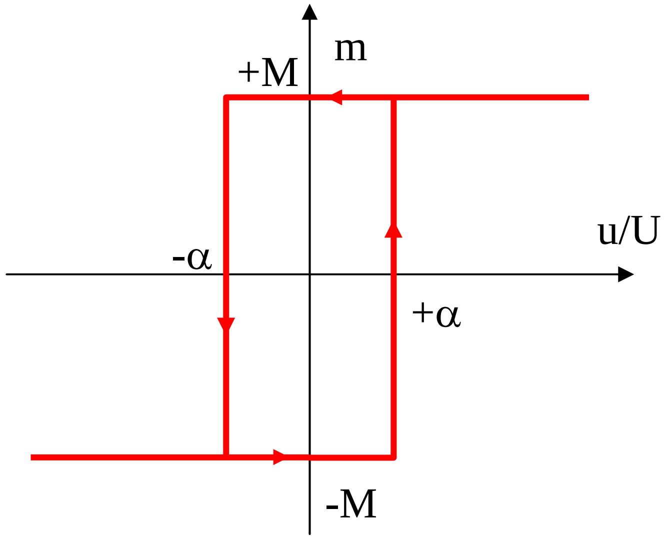


$$N(\alpha)=\begin{cases}0 & \alpha\geq 1\\ \frac{k}{2}\left[1-\Phi(2\alpha-1)\right]-j\frac{4k\alpha}{\pi}(1-\alpha) & \alpha<1\end{cases}$$

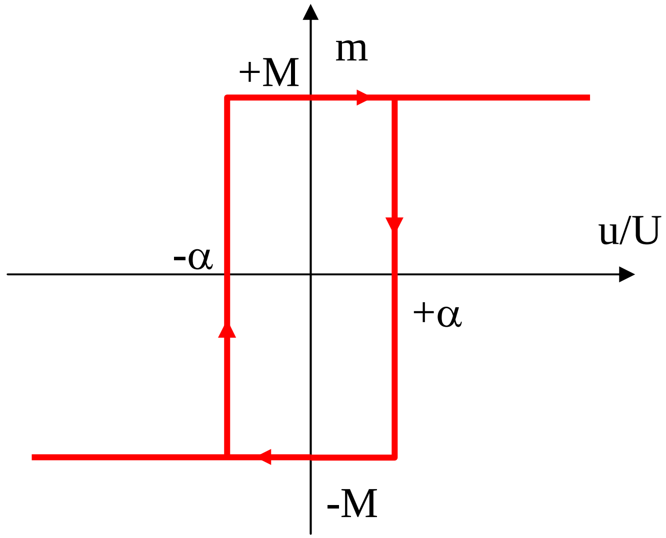


$$N(\alpha)=\begin{cases}0 & \alpha\geq 1\\ \frac{k}{2}\left[1-\Phi(2\alpha-1)\right]+j\frac{4k\alpha}{\pi}(1-\alpha) & \alpha<1\end{cases}$$

Describing Function – *Computation*

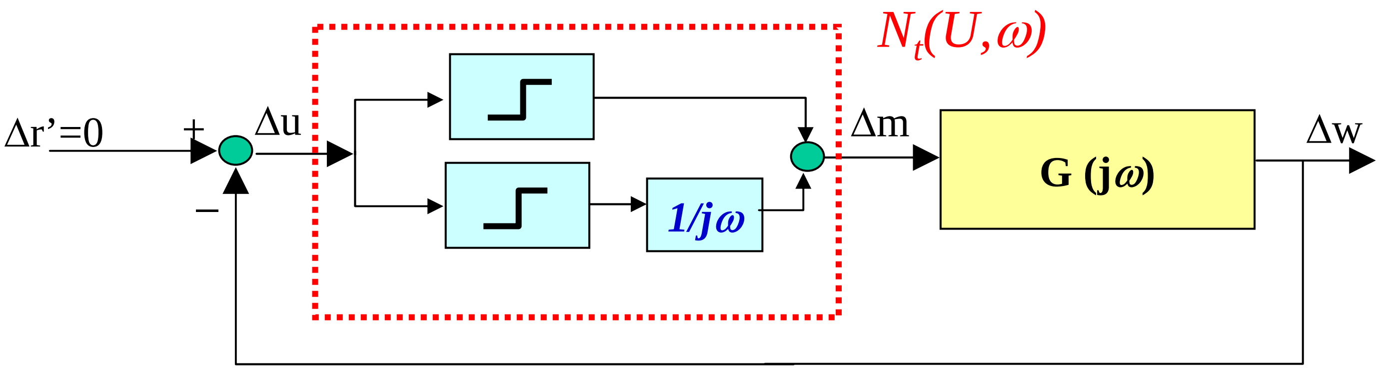


$$N(U,\omega)=\begin{cases}0 & \alpha\geq 1\\ \dfrac{4M}{\pi U}\sqrt{1-\alpha^2}-j\dfrac{4M\alpha}{\pi U} & \alpha<1\end{cases}$$



$$N(U,\omega)=\begin{cases}0 & \alpha\geq 1\\ \dfrac{4M}{\pi U}\sqrt{1-\alpha^2}+j\dfrac{4M\alpha}{\pi U} & \alpha<1\end{cases}$$

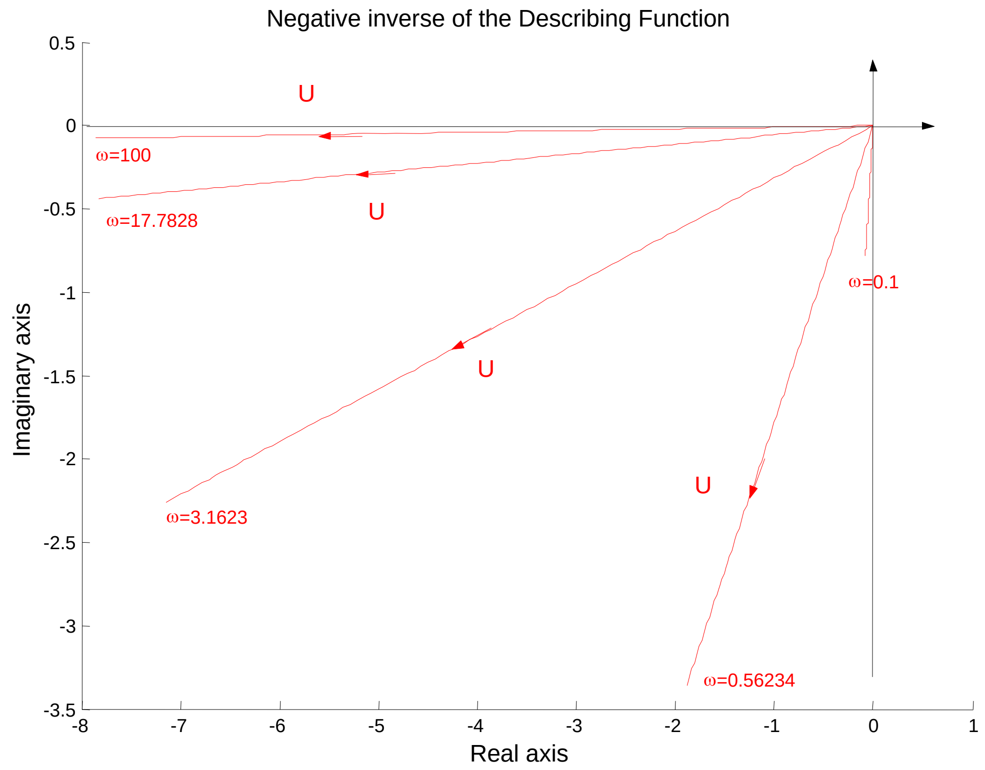
Describing Function – *Computation*



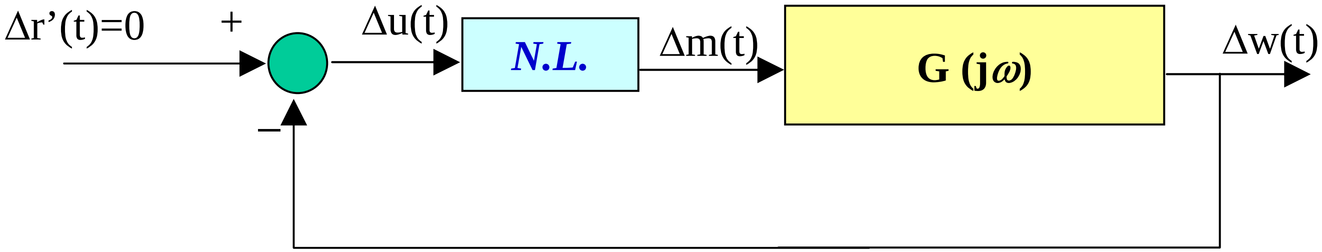
$$N_t(U) = N_1(U) + N_1(U)\frac{1}{j\omega}$$

$$N(U,\omega) = \frac{4M}{\pi U} - j\frac{4M}{\pi U\omega}$$

$$\frac{-1}{N(U,\omega)} = -\frac{\pi U\omega}{4M\left(1+\omega^2\right)}(\omega + j)$$



Stability of limit cycles



If the block *N.L.* is a pure constant *k*, the stability of the feedback system can be performed by means of the Nyquist criterion, which gives the number of roots with positive real roots of the Harmonic Balance Equation

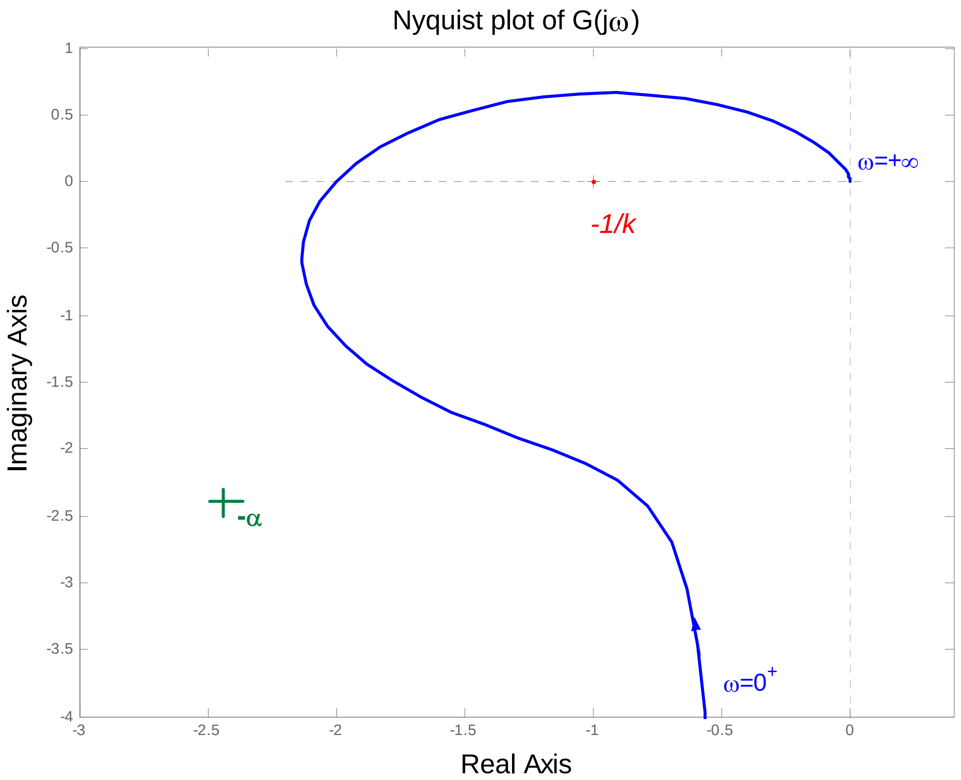
$$1 + G(j\omega)\frac{1}{k} = 0$$

The Nyquist criterion looks at the relative position of the transfer function $G(j\omega)$ with respect to the point $(-1/k,\ 0)$ in the complex plain.
By extension the criterion can be applied to any point $-\alpha$ of the complex plain with reference to the Harmonic Balance Equation

$$1 + G(j\omega)\alpha = 0, \quad \alpha \in \mathbb{C}$$

Stability of limit cycles

If the transfer function $G(j\omega)$ represents a stable system, the reduced Nyquist criterion can be applied, i.e. the closed loop stability can be stated if the reference point $-\alpha$ in the complex plain remains on the left-side when running along the Nyquist plot of $G(j\omega)$ from $\omega=0^+$ to $\omega=+\infty$.



$$G(j\omega)=\frac{1}{j\omega\left((j\omega)^2+0.5j\omega+1\right)}$$

The closed loop is not stable with respect to the point $(-1/k,0)$.

The closed loop is stable with respect to the point $-\alpha$

Stability of limit cycles

Can the generalization of the Nyquist criterion be used to analyse the stability of a limit cycle?

YES, *it can*

What does it mean that a limit cycle is stable?

If a perturbation of the magnitude of the periodic oscillation occurs, it tends to the original value as time passes.

How can a limit cycle be considered from the Nyquist criterion point of view?

It is a marginally stable condition.

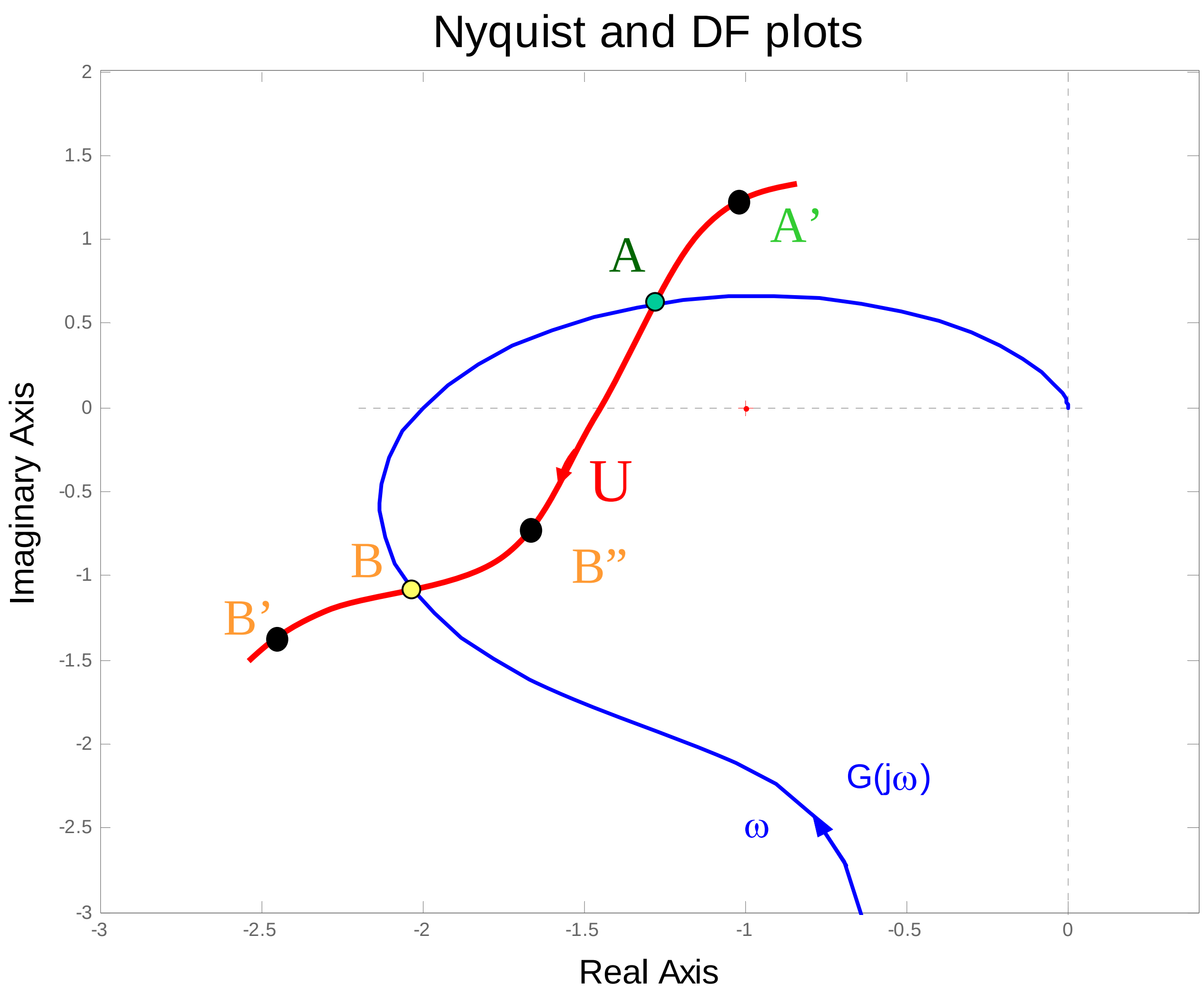
How can the magnitude of a limit cycle be represented?

By a point in the Describing Function plot.

Stability of limit cycles

Points **B**, **B'**, **B"**, **A** and **A'** represent possible magnitudes of the oscillation.

A and **B** represent two possible limit cycles



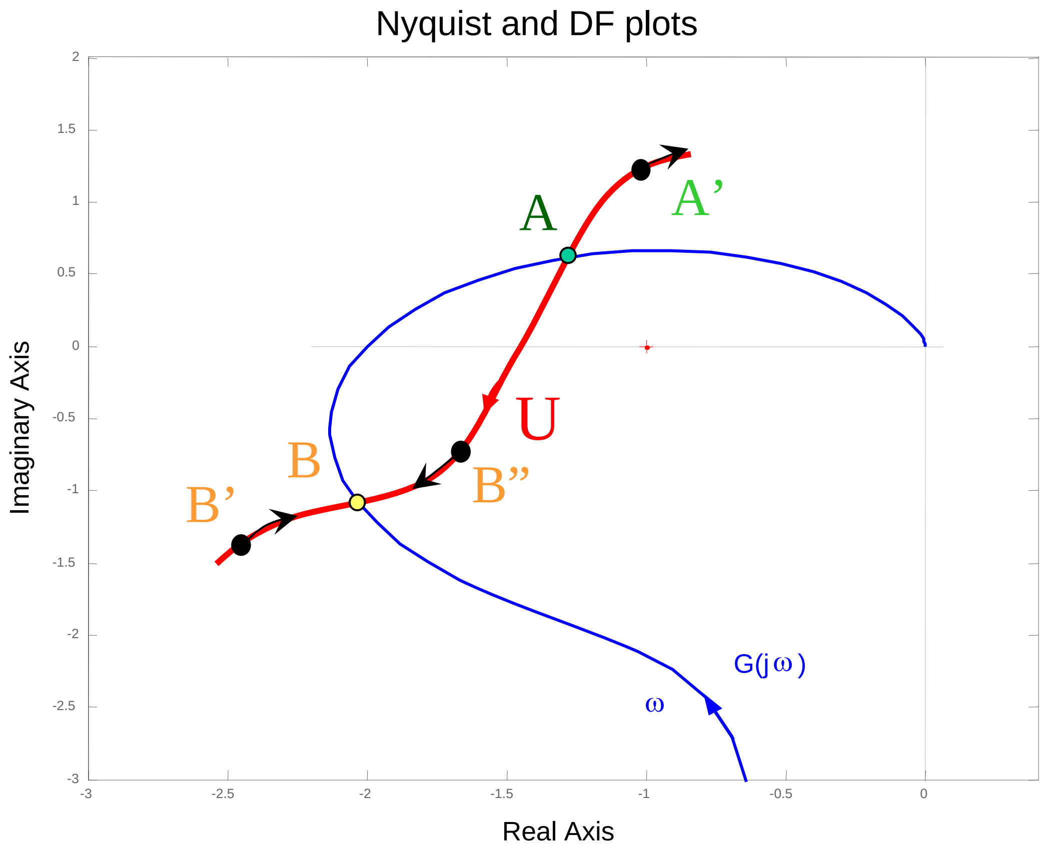
Stability of limit cycles

Applying the reduced Nyquist criterion with respect to:

point **B’:** oscillations tend to decrease (stable system)

point **B”:** oscillations tend to increase (unstable system)

point **A’:** oscillations tend to decrease (stable system)



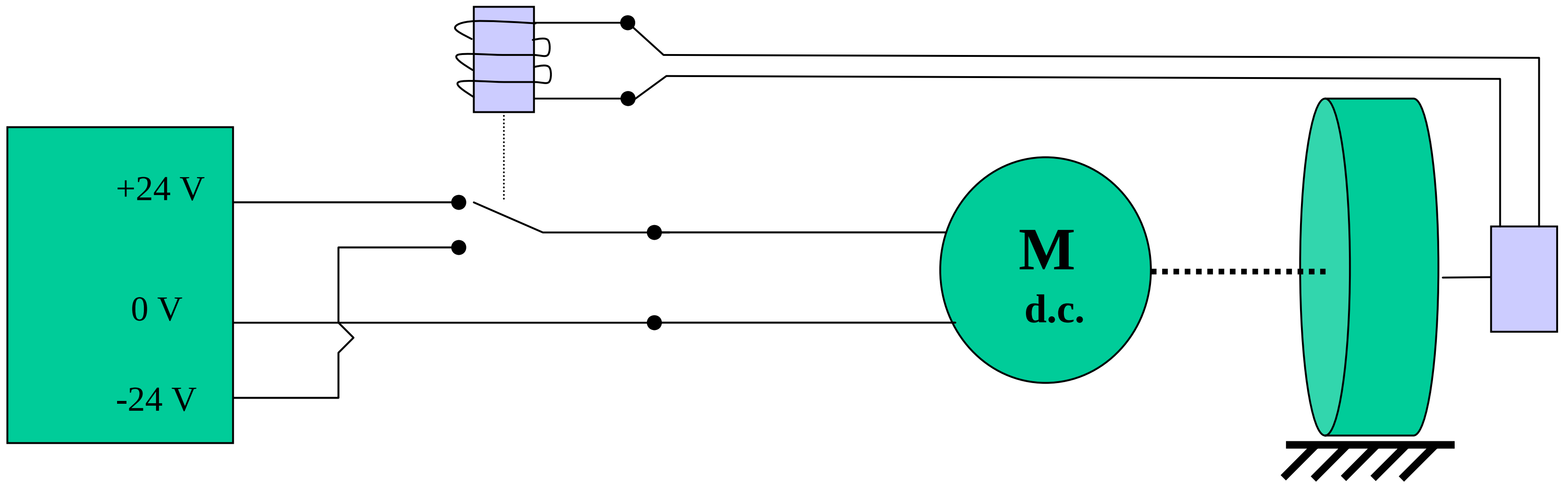
B: Stable limit cycle

A: Unstable limit cycle

The linear system G(jω) is assumed to be stable in order to apply the reduced Nyquist criterion

DF Analysis – *Example of application*

Consider a DC motor with permanent magnets



The position of the motor shaft is measured by means of a rotational variable resistance.

The voltage on the rotational resistance drives the position of a relay that switches the motor supply voltage between +/- 24 V d.c.

DF Analysis – *Example of application*

The linear approximate model of a DC motor is the following

$$v_r(t) = R_r i_r(t) + L_r \frac{di_r(t)}{dt} + e(t)$$

$$J\frac{d\omega(t)}{dt} = C_{em}(t) + B\omega(t)$$

$$\omega = \frac{d\vartheta(t)}{dt}$$

$$e(t) = k_e \omega(t)$$

$$C_{em}(t) = k_t i_r(t)$$

$$J = J_m + J_l$$

$$B = B_m + B_l$$

$$v_r(t) = \begin{cases} +24 & \vartheta < \vartheta_d \\ -24 & \vartheta \geq \vartheta_d \end{cases}$$

R_r: rotor resistance
L_r: rotor inductance
k_e: voltage feedback constant
k_t: torque constant
J_m: motor inertia
J_l: load inertia
B_m: motor friction coefficient
B_l: load friction coefficient

v_r: rotor supply voltage
i_r: rotor wound current
C_{em}: electromagnetic torque
ω: rotational speed
θ: shatf angular position

DF Analysis – *Example of application*

The motor transfer functions are the following

$$W_\omega(s) = \frac{\Omega(s)}{V_s(s)} = \frac{k_t}{(sL_a + R_a)(sJ + B) + k_t k_e}$$

$$W_\vartheta(j\omega) = \frac{\Theta(j\omega)}{V_r(j\omega)} = \frac{k_t}{j\omega((j\omega L_a + R_a)(j\omega J + B) + k_t k_e)} = \Re(j\omega) + j\Im(j\omega)$$

$$= - \frac{k_t(R_a J + L_a B)}{\omega^2(R_a J + L_a B)^2 + (R_a B + k_t k_e - \omega^2 L_a J)^2}$$

$$- j\frac{k_t(R_a B + k_t k_e - \omega^2 L_a J)}{\omega\left[\omega^2(R_a J + L_a B)^2 + (R_a B + k_t k_e - \omega^2 L_a J)^2\right]}$$

DF Analysis – *Example of application*

Taking into account the parameters of the motor and of the load

R=0.4; % rotor resistance

L=0.001; % rotor inductance

ke=0.3; % voltage feedback constant

kt=0.3; % torque constant

Jm=0.01; % motor inertia

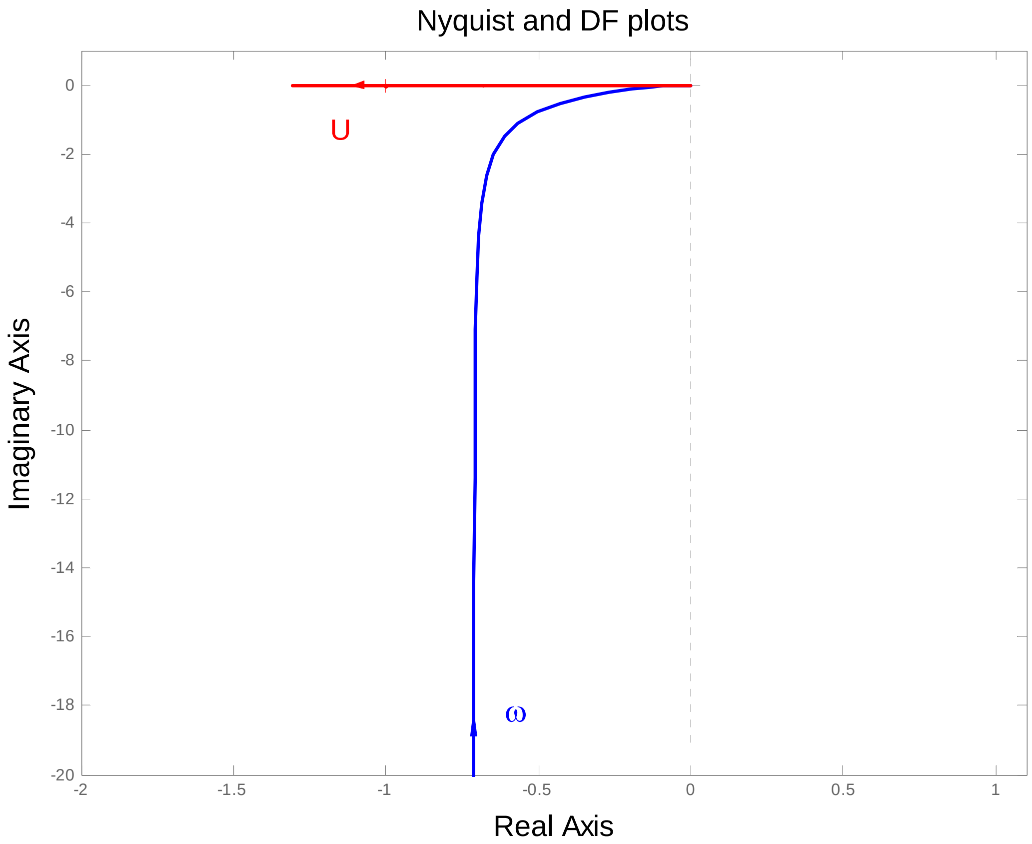
Jl=0.09 % load inertia

Bm=0.05; % motor friction coefficient

Bl=0.05; %load friction coefficient

J=Jm+Jl;

B=Bm+Bl;



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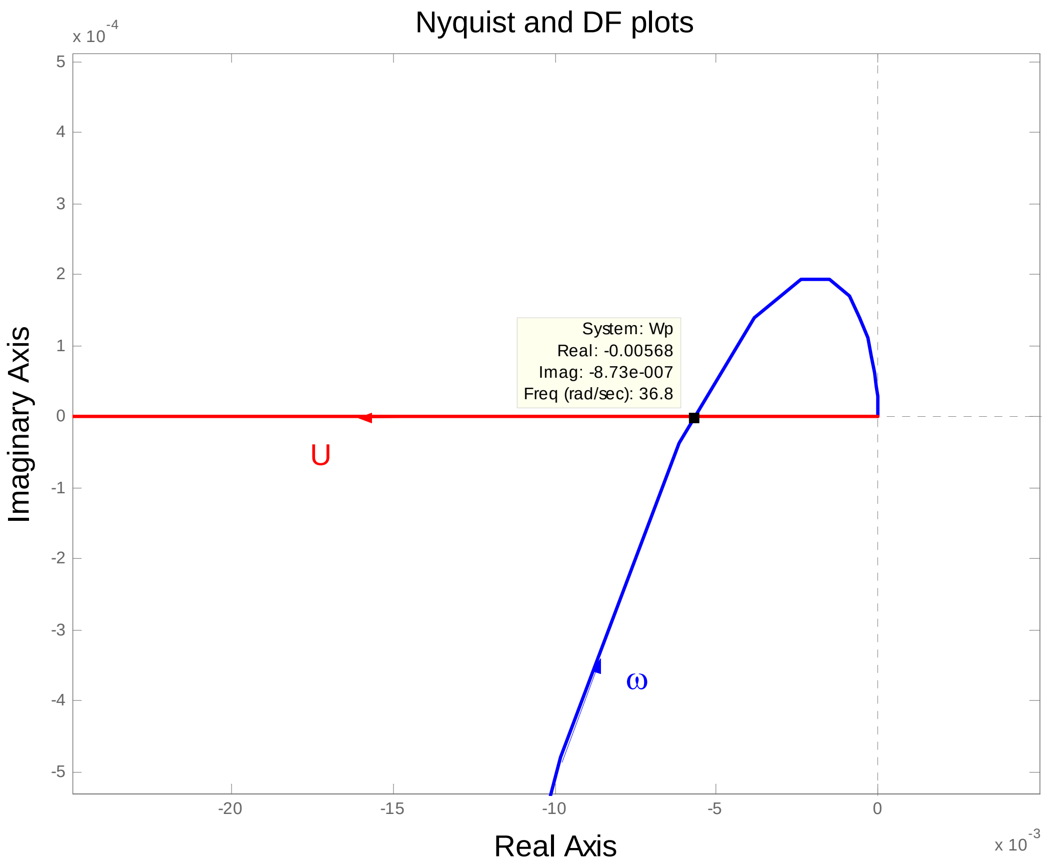

$$\omega\big|_{\Im(W_p(j\omega))=0} = \omega_{cr} = \sqrt{\frac{R_a B + k_t k_e}{L_a J}} = 36.056\ \text{rad/s} \qquad \overline{\text{U}} = -\frac{4\text{M}}{\pi}\cdot\Re\left(W_p\left(j\omega_{cr}\right)\right) = 0.1759\ \text{rad}$$

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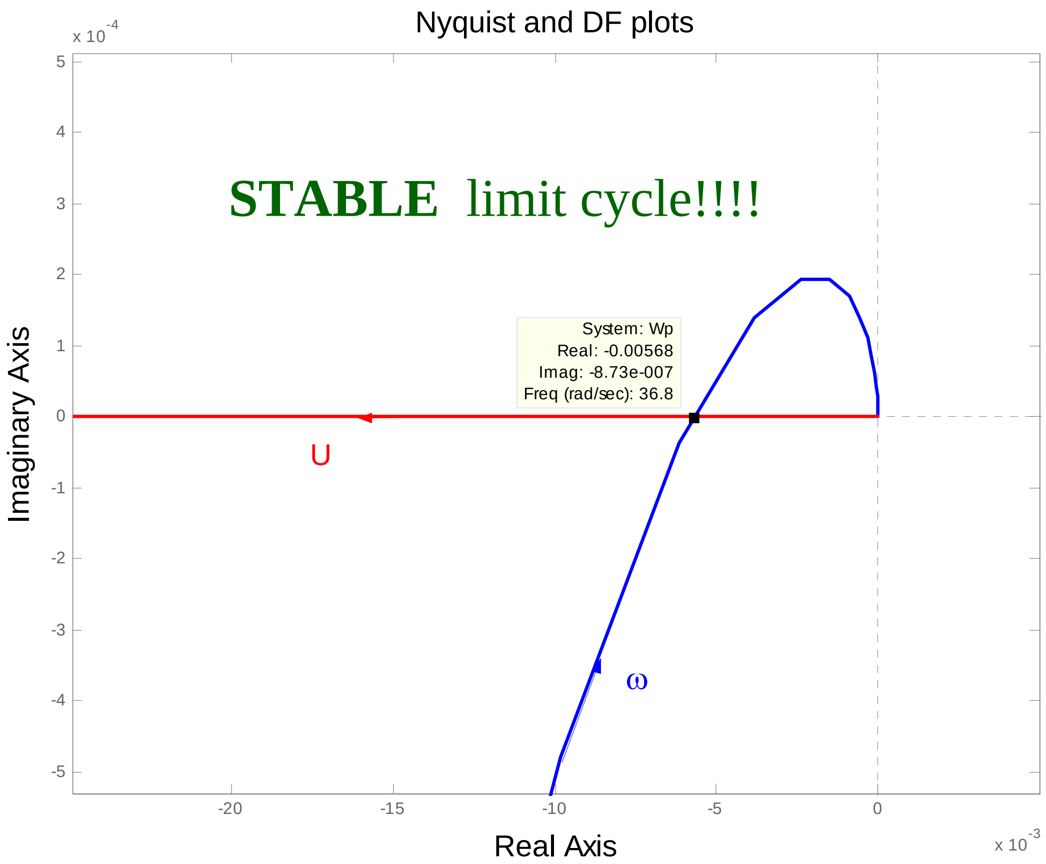

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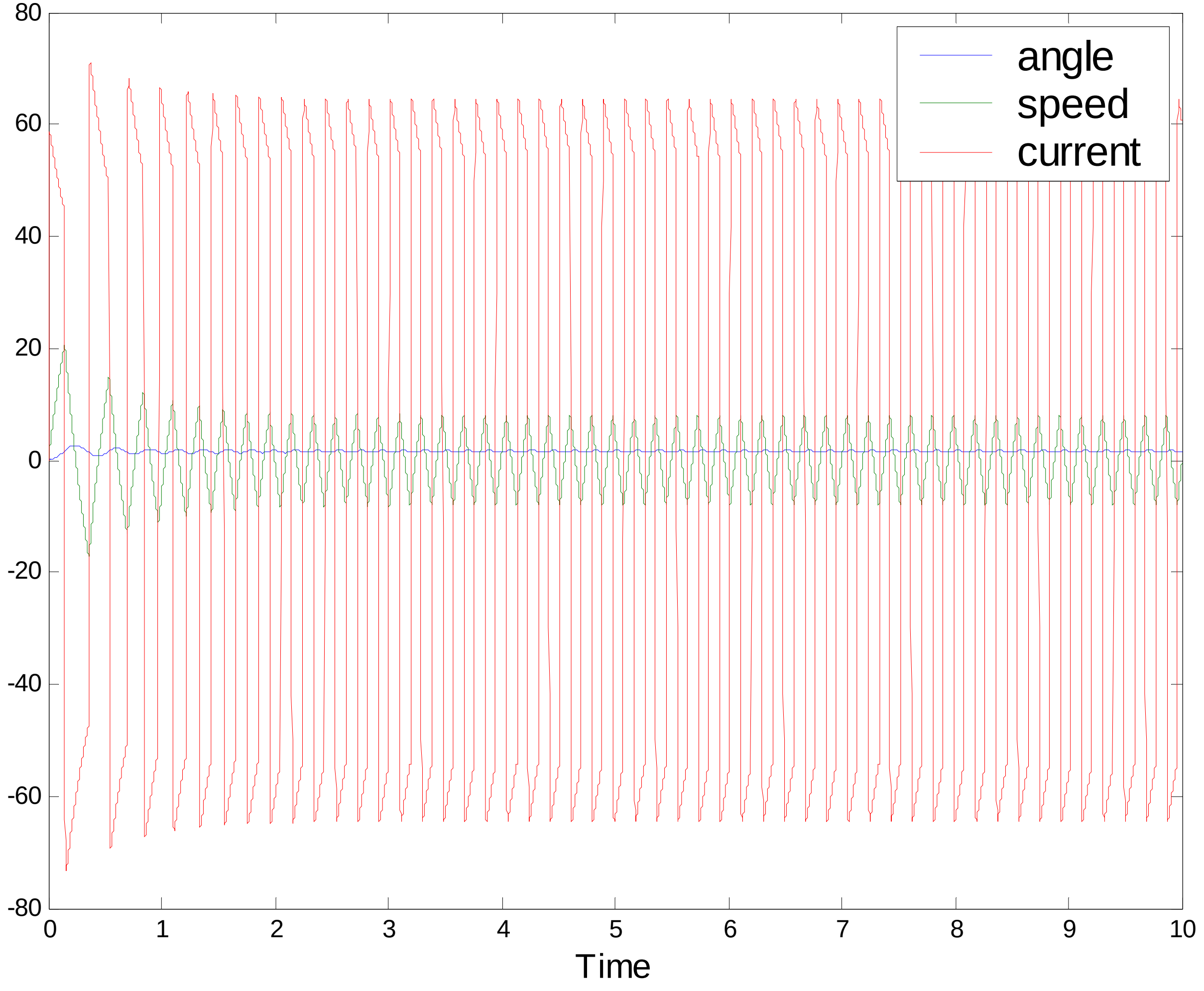
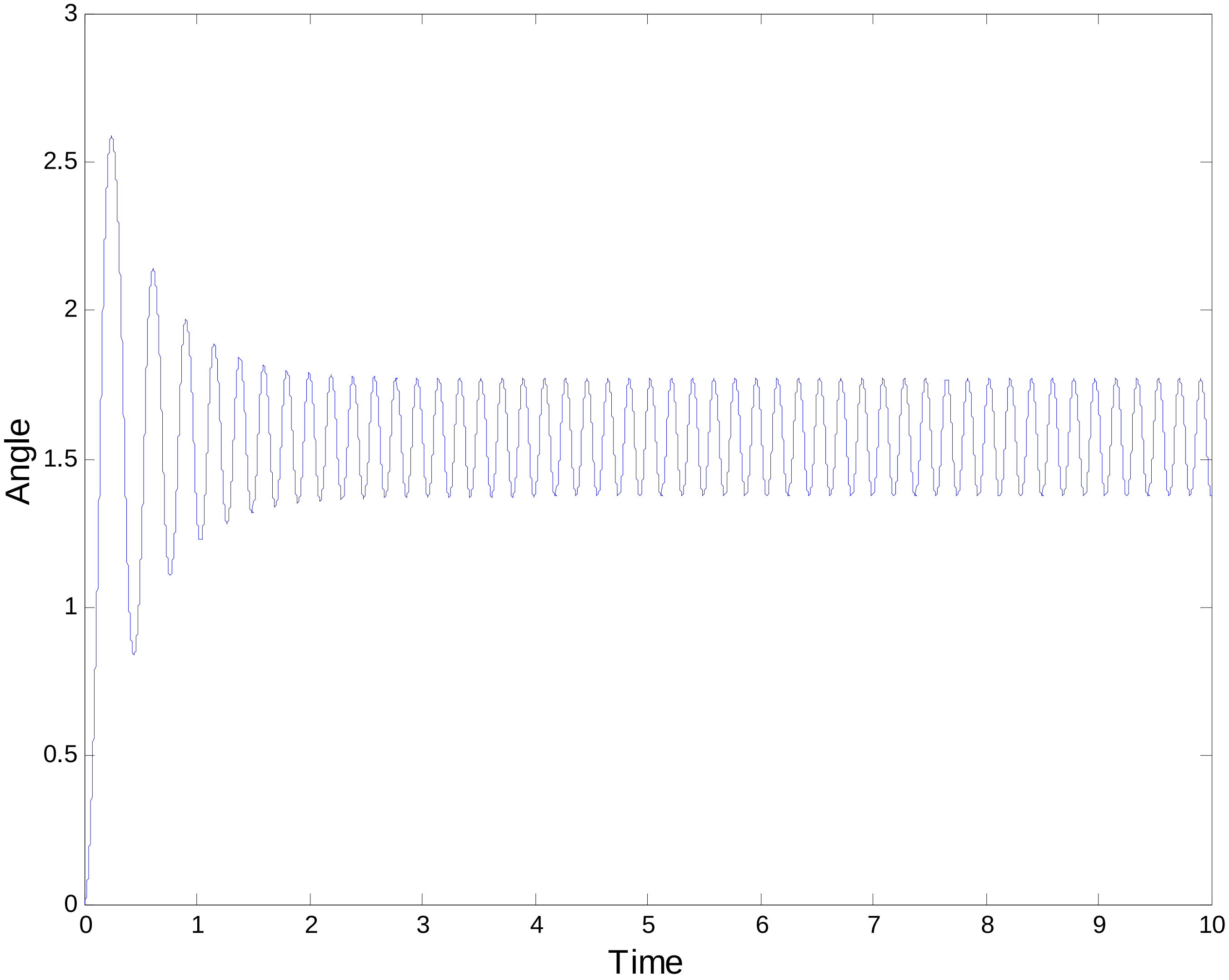
DF Analysis – *Example of application*

The system presents a periodic steady-state oscillation



DF Analysis – *Example of application*

The system presents a periodic steady-state oscillation



DF Analysis – *Example of application*

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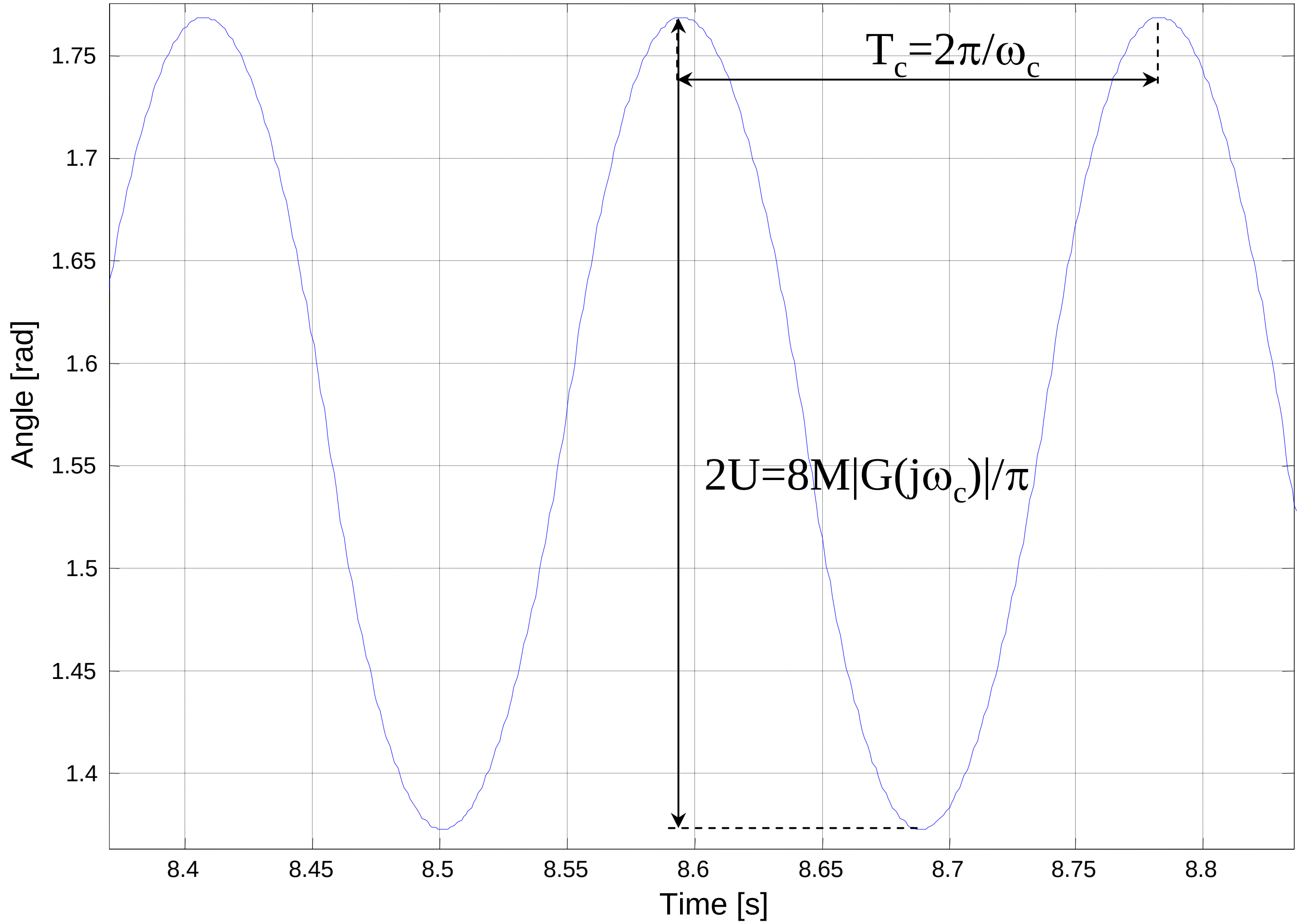
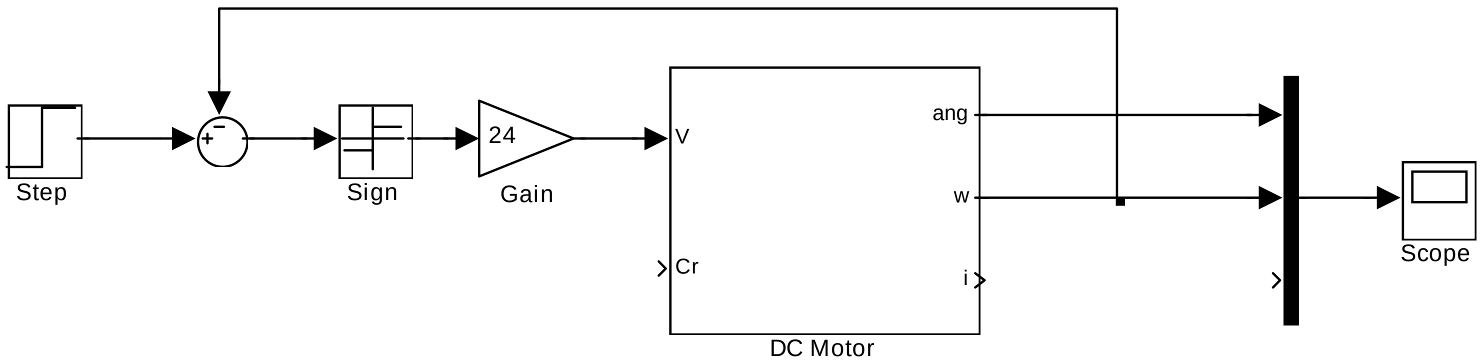


DF Analysis – *Example of application*

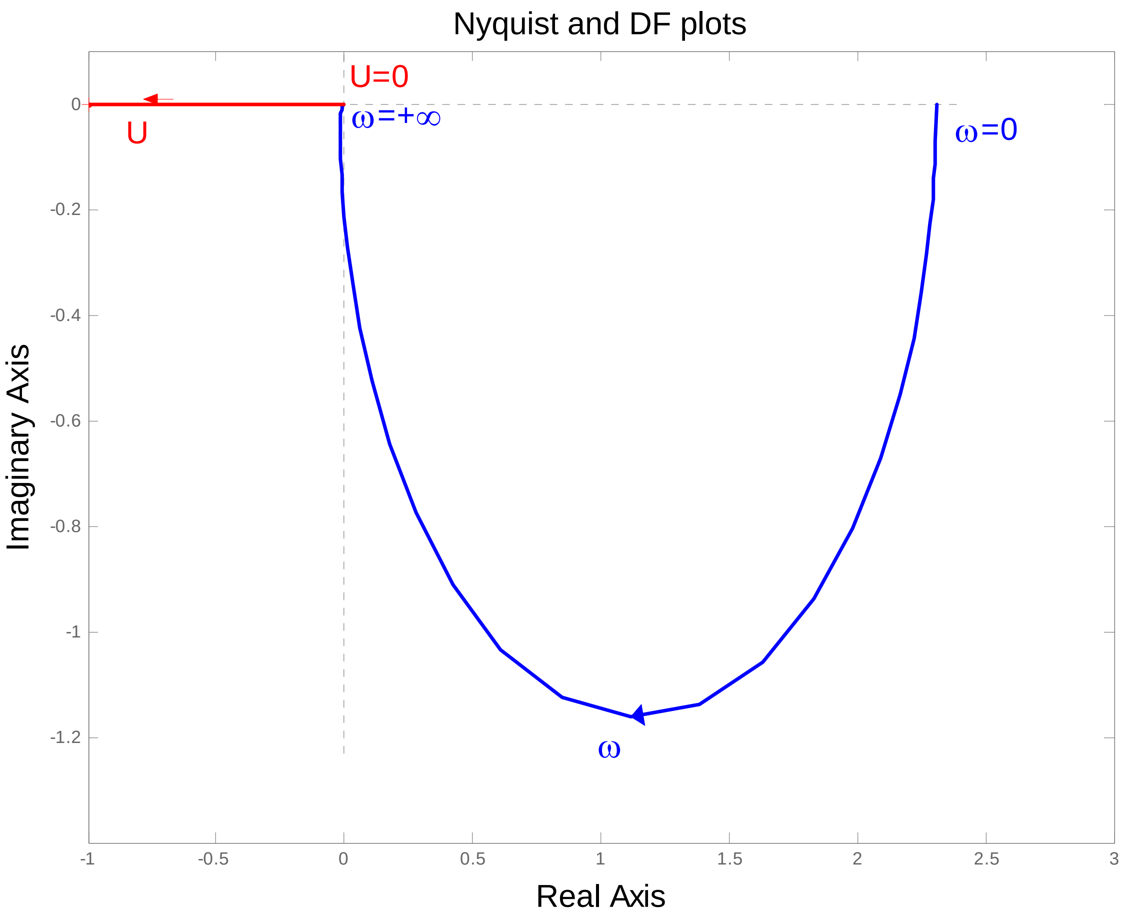
What does it happen if the same control law is applied to the speed control problem?



$$W_\omega(s) = \frac{k_t}{(sL_a + R_a)(sJ + B) + k_t k_e}$$

The linear plant is characterised by an all-pole transfer function with relative degree two, therefore there is no contact point between the Nyquist plot and the real negative axis, but the origin at $\omega=+\infty$ (corresponding to U=0 in the DF plot)

DF Analysis – *Example of application*

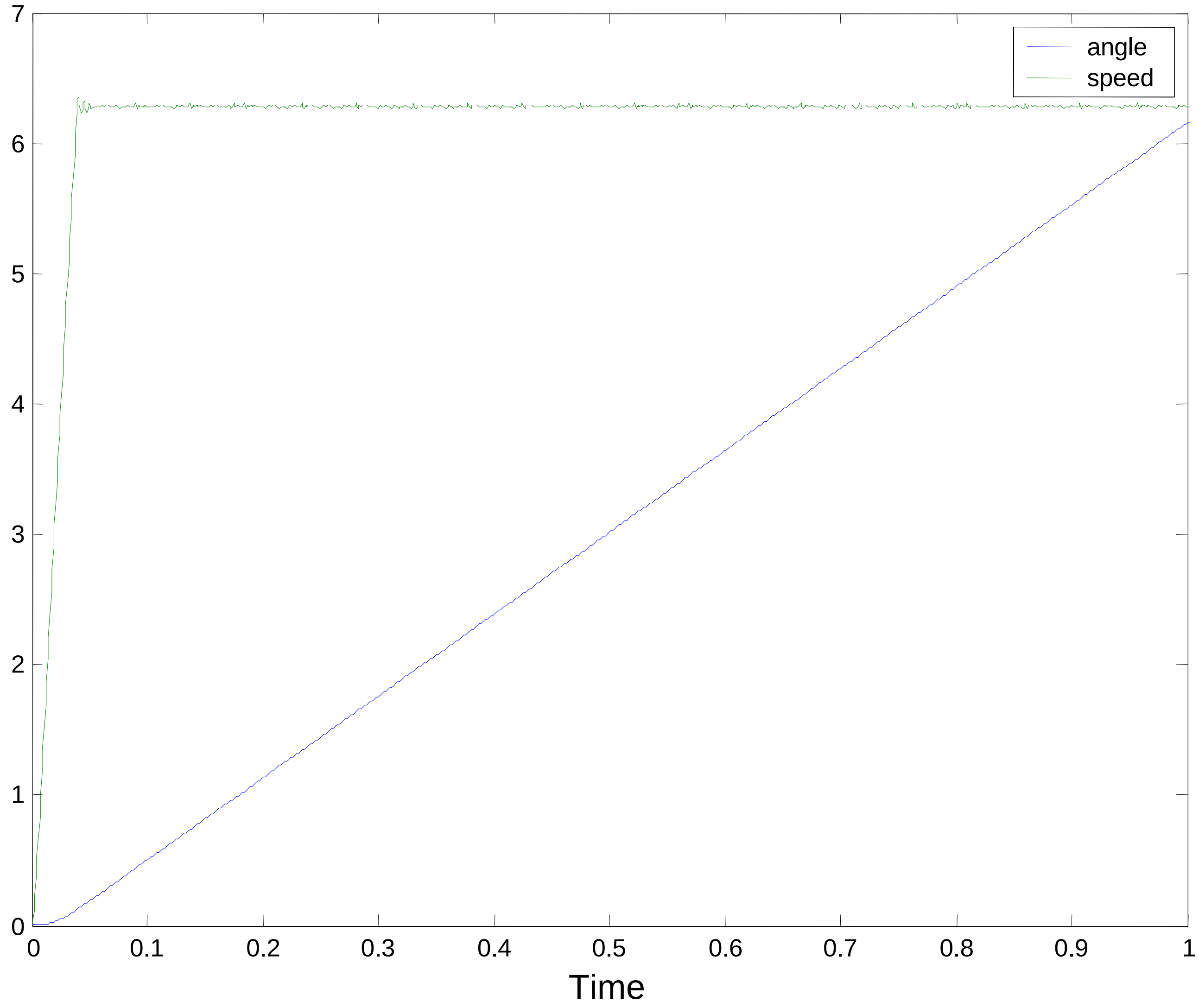


The Nyquist plot of the linear system is **tangent** to the negative reciprocal of the DF at the origin, i.e., U=0 and ω=+∞

Maybe a sliding mode behaviour is established asymptotically!?

DF Analysis – *Example of application*

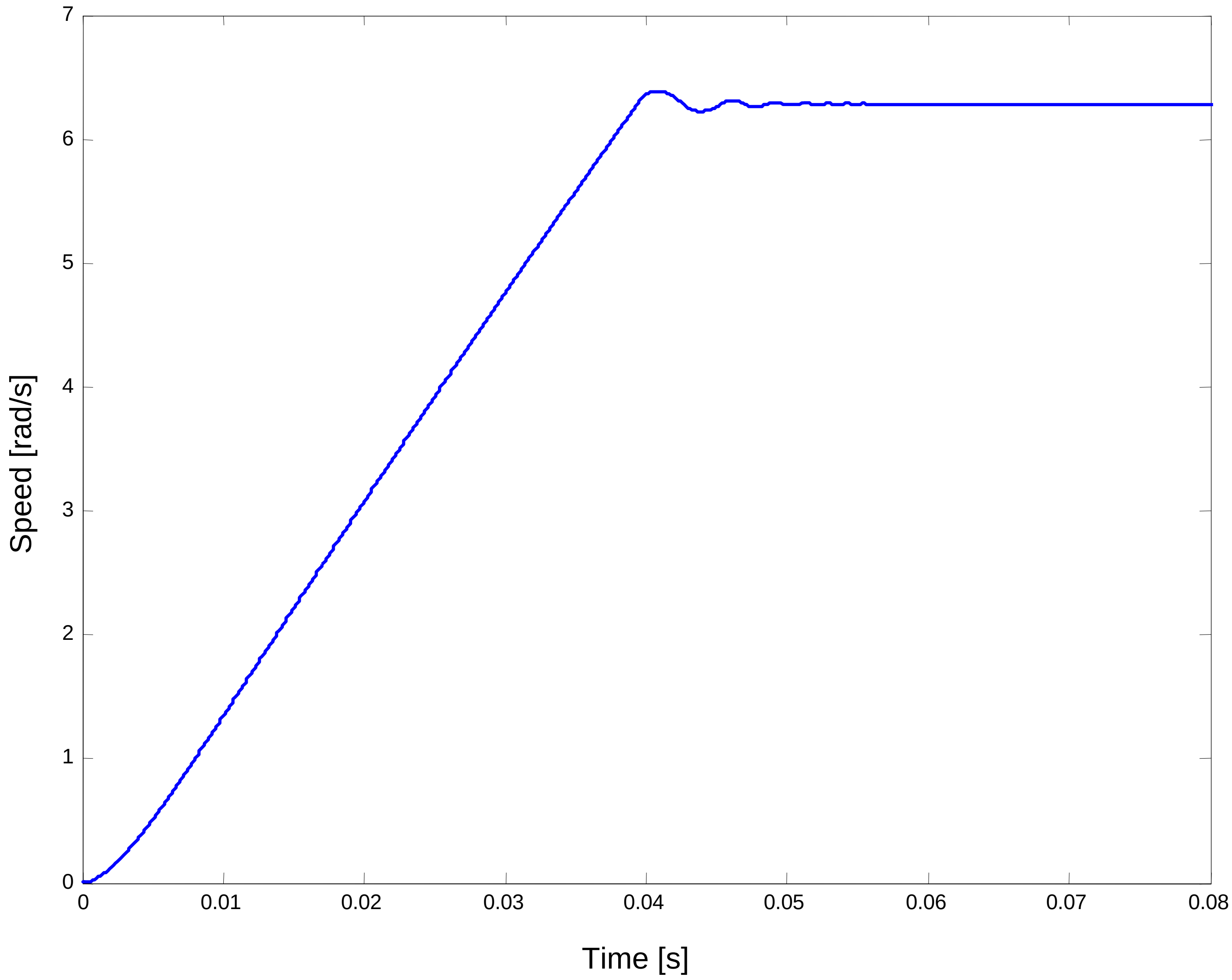
Possibly a sort of dead-bit control, or finite time stabilisation seams to appear



Can it be a sliding mode behaviour?

DF Analysis – *Example of application*

By reducing the integration step of the simulation it is apparent that the stabilisation is asymptotic

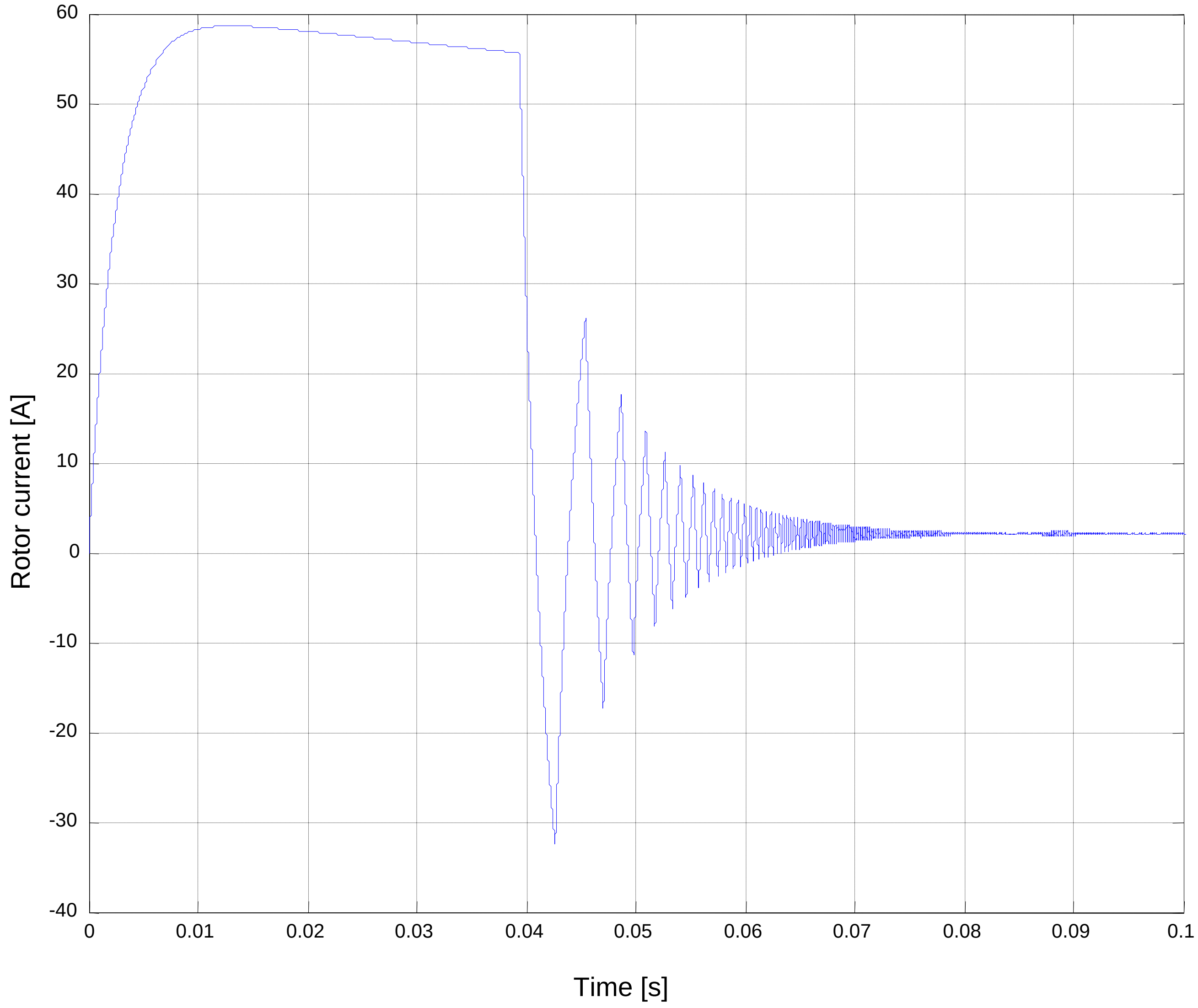


Can it be a sliding mode behaviour?

DF Analysis – *Example of application*

By reducing the integration step of the simulation it is apparent that the stabilisation is asymptotic

The rotor current tends to a constant value, i.e. an asymptotic (*2nd order*) sliding sliding mode appears



DF Analysis – *Example of application*

The DC motor is a second order system whose state variables are the rotor speed and current

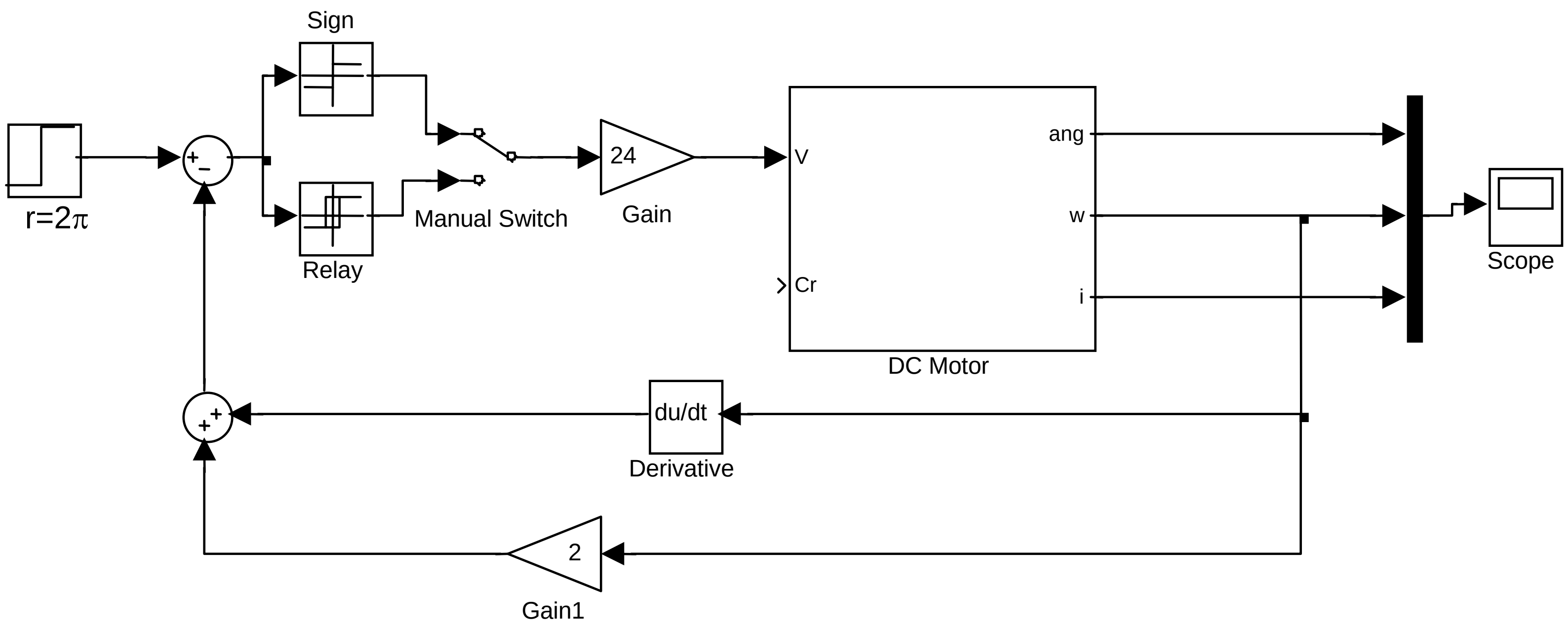
$$\begin{aligned}\frac{d\omega(t)}{dt} &= \frac{B}{J}\omega(t)+\frac{k_t}{J}i_r(t)\\ \frac{di_r(t)}{dt} &= -\frac{k_e}{L_r}\omega(t)-\frac{R_r}{L_r}i_r(t)+\frac{1}{L_r}v_r(t)\end{aligned} = \begin{bmatrix} \frac{B}{J} & \frac{k_t}{J} \\ -\frac{k_e}{L_r} & -\frac{R_r}{L_r} \end{bmatrix}\begin{bmatrix}\omega(t)\\ i_r(t)\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{L_r}\end{bmatrix}v_r(t)$$

$$s(t)=\frac{B}{J}\omega(t)+\frac{k_t}{J}i_r(t)+2\omega(t)=\begin{bmatrix}2+\frac{B}{J} & \frac{k_t}{J}\end{bmatrix}\begin{bmatrix}\omega(t)\\ i_r(t)\end{bmatrix}$$

The system transfer function has a zero in –2, and if the system output is steared to zero in a finite time, than the system behaves as a first order system with time constant τ=0.5

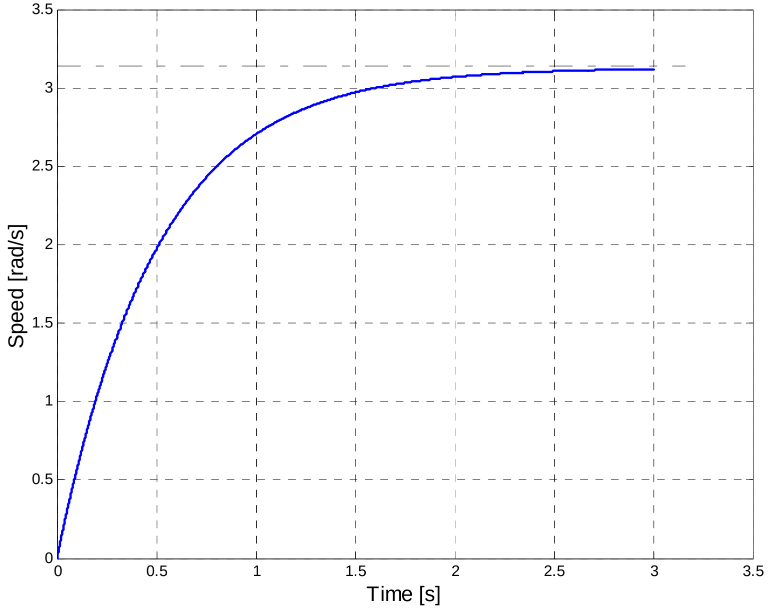
DF Analysis – *Example of application*

A MATLAB-Simulink scheme is the following



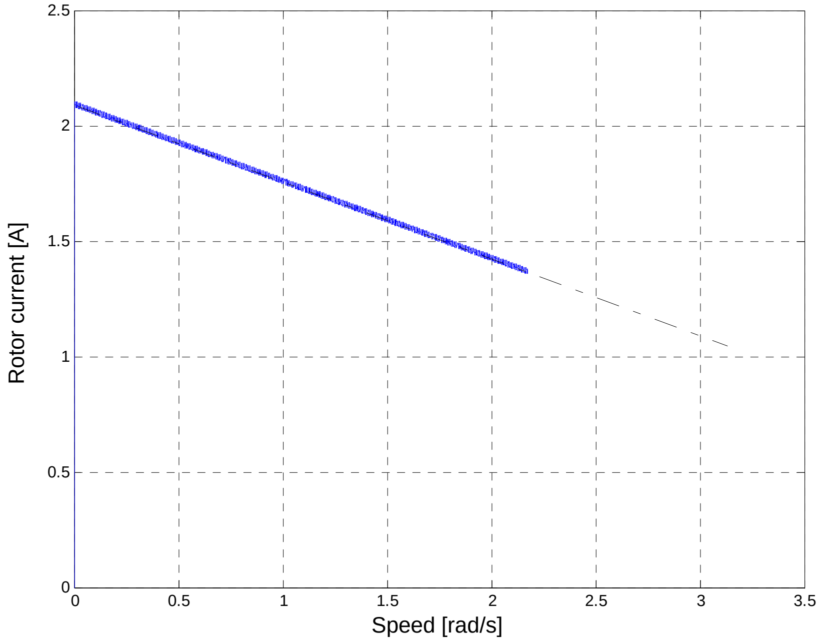
Take care that because of the feedback, the new closed loop gain (assuming the rotor velocity as the output) will be 1/2

DF Analysis – *Example of application*

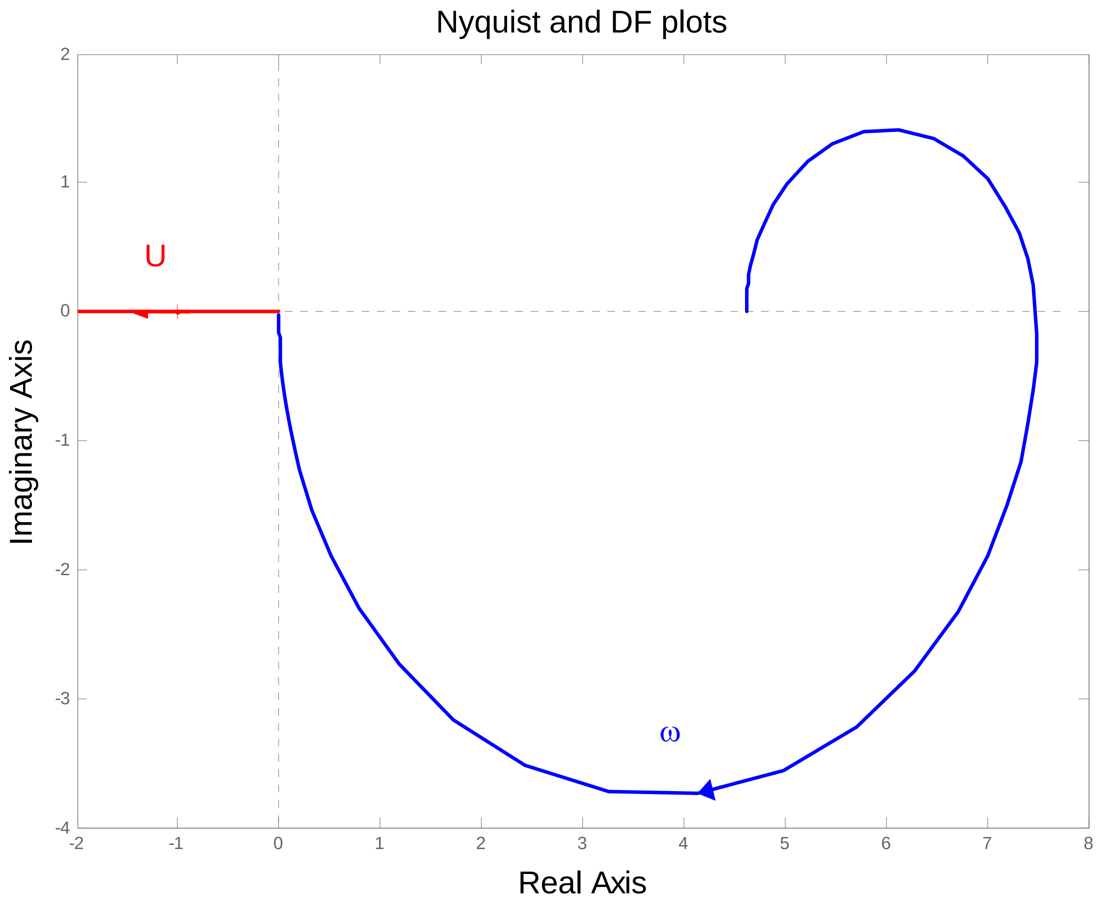


It is apparent that the shaft speed tends to the steady state value π as a first order system with τ = 0.5 s

Apart from a very short transient, the state trajectory tends to the steady-state values sliding on a linear manifold of the state space



DF Analysis – *Example of application*



The Nyquist plot of the linear system **cross** the negative reciprocal of the DF at the origin, i.e., U=0 and $\omega=+\infty$

A sliding mode behaviour is established in a finite time

DF Analysis & Sliding Modes

Sliding modes are characterised by infinite frequency of proper ideal switching devices

Most sliding mode controllers use a sign function in the controller, i.e., an ideal relay, which can be approximately represented by its Describing Function

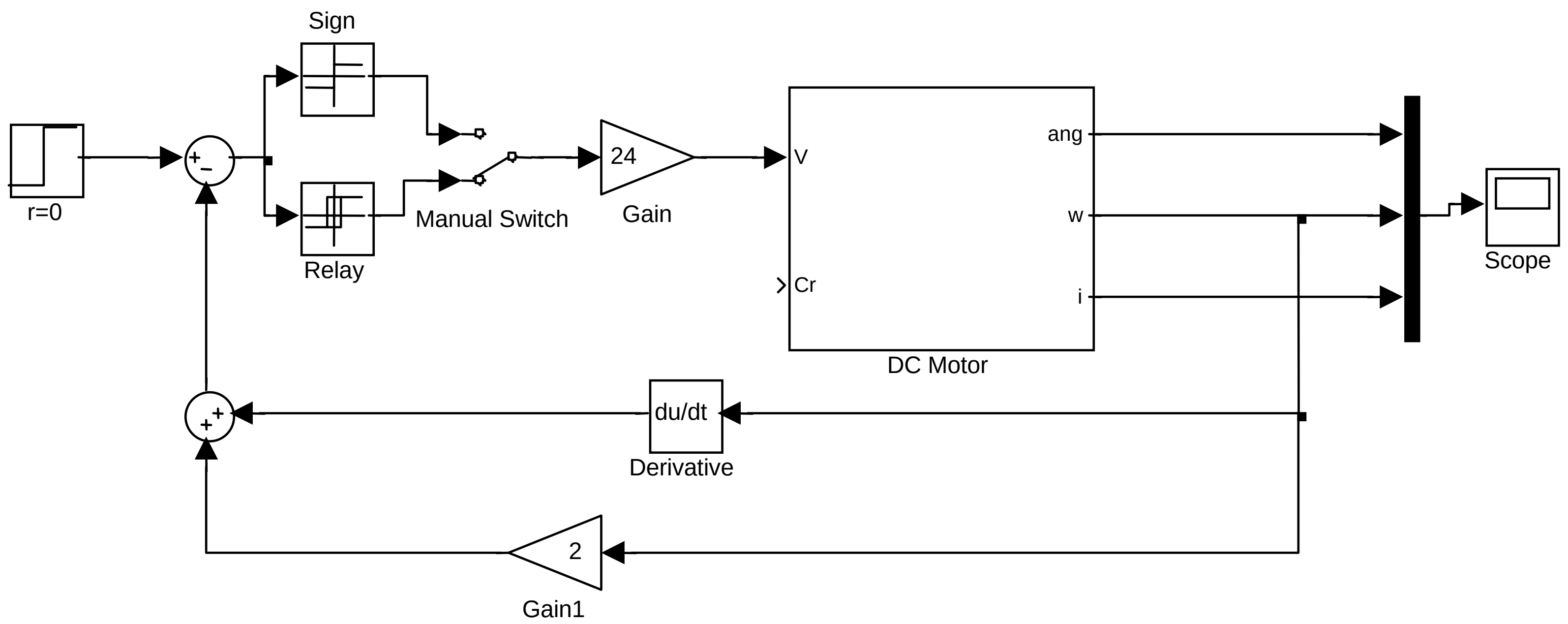
By the example it is apparent that a sliding mode behaviour is established in a finite time if the Nyquist plot of **cross** the inverse negative of the describing function at the point ($U=0,\omega=+\infty$)

It can also be derived that a sliding mode behaviour is established asymptotically if the Nyquist plot of **is tangent to** the inverse negative of the describing function at the point ($U=0,\omega=+\infty$)

TAKE CARE: the Describing function is an approximate tool

DF Analysis & Sliding Modes

Substitute the ideal relay with a hysteretic one with β=1

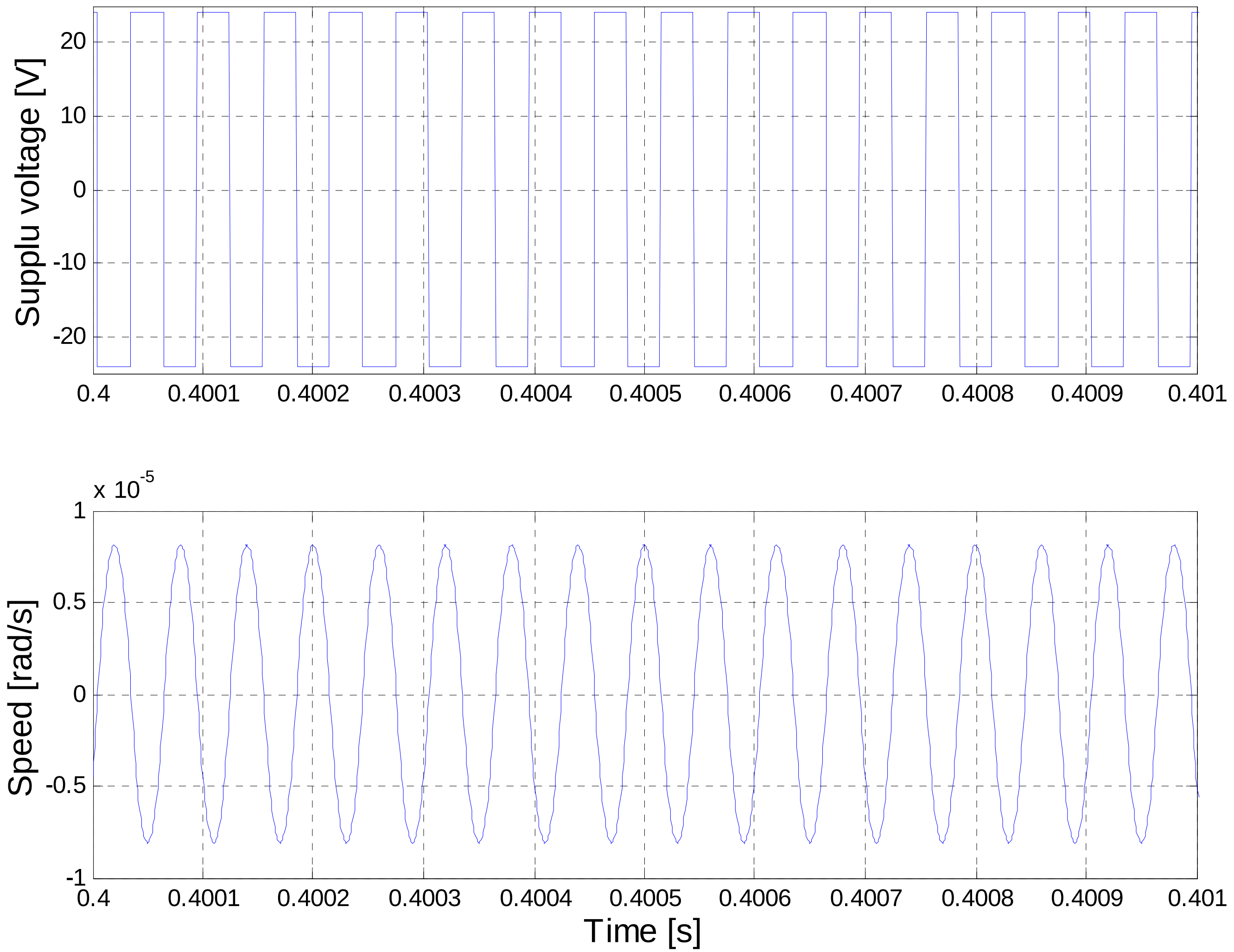


It is apparent that an ideal sliding mode cannot appear because of the switching delay

DF Analysis & Sliding Modes

The limit cycle is apparent, with a period $T_{lc} \cong 0.6$ ms

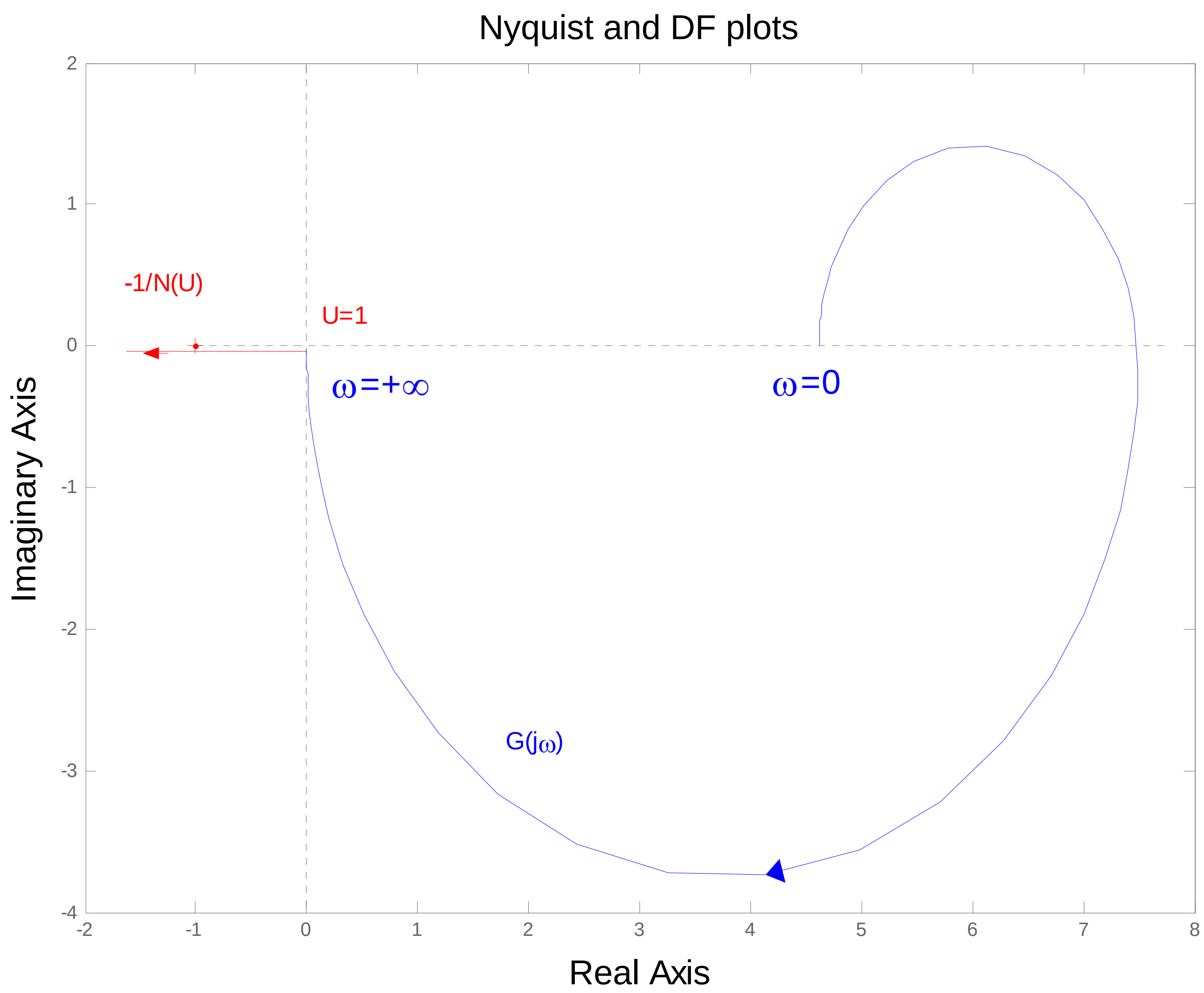
The magnitude is very small because of the low-pass filter property of the motor transfer function



DF Analysis & Sliding Modes

The Nyquist plot of the linear system has no common point with the negative inverse of the Describing Function of the hysteretic relay

Describing function analysis can be useful in sliding mode systems when a common point is present



DF Analysis & Sliding Modes

Most effective use of the Describing Function approach to sliding modes….

⇓

Analysis of the characteristics of a chattering behaviour due to unmodelled dynamics of sensors and/or actuators

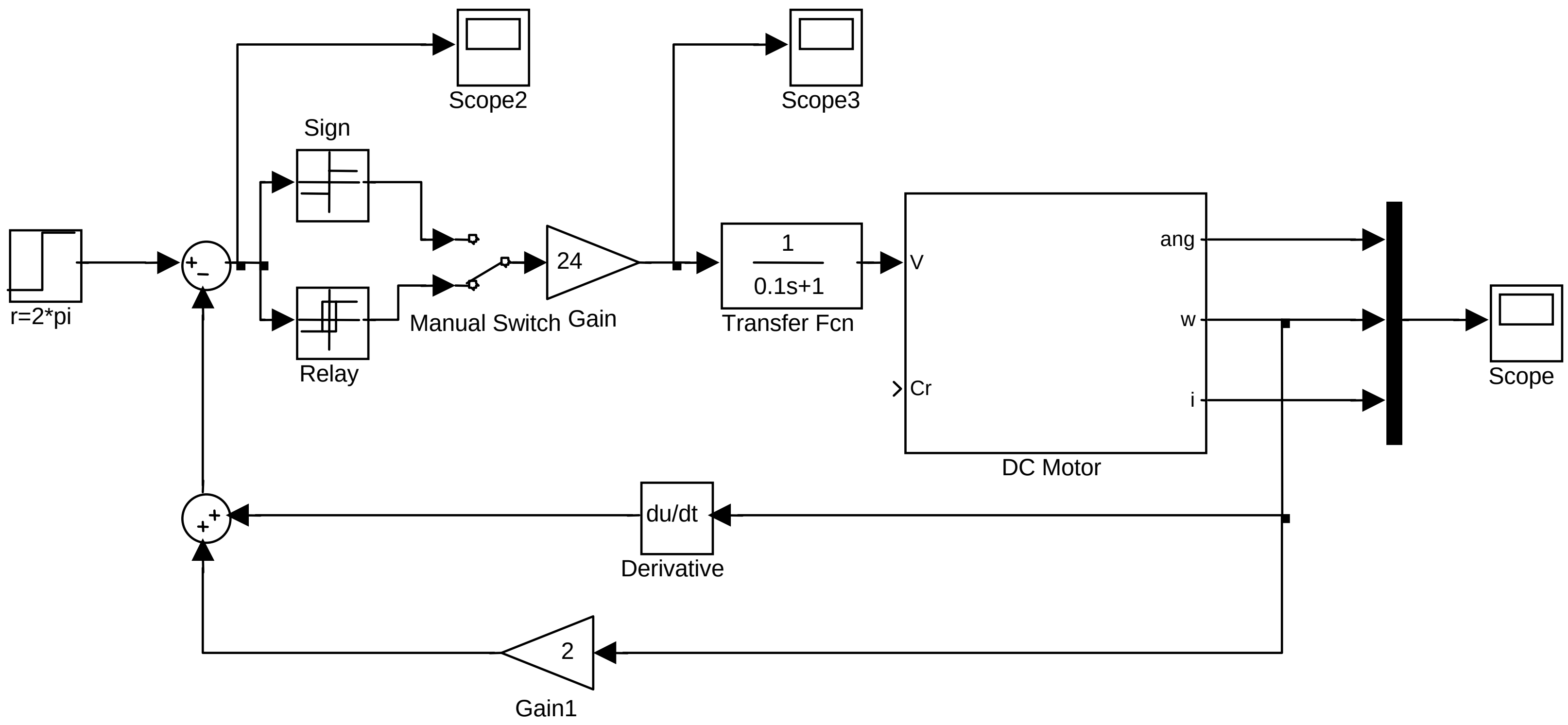
What is chattering? *It appears as oscillations of the system variables, whose magnitude is related to the influence of the neglected dynamics on the system bandwidth*

⇓

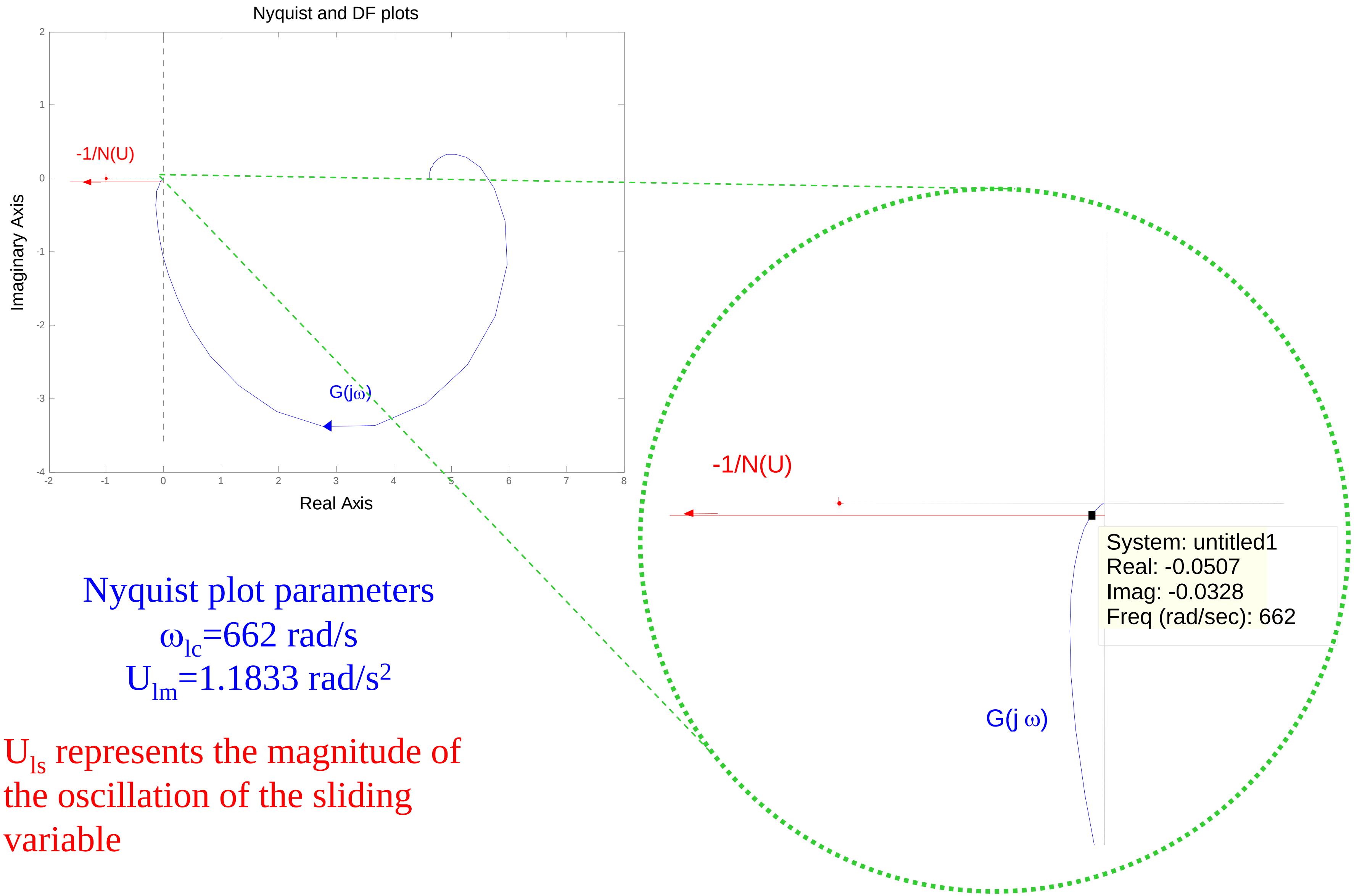
It is very close to a limit cycle

DF Analysis & Sliding Modes

Consider the motor drive as a hysteretic switching device plus a time constant τ_a=0.1 s



DF Analysis & Sliding Modes



Nyquist plot parameters

ω_{lc}=662 rad/s

U_{lm}=1.1833 rad/s^2

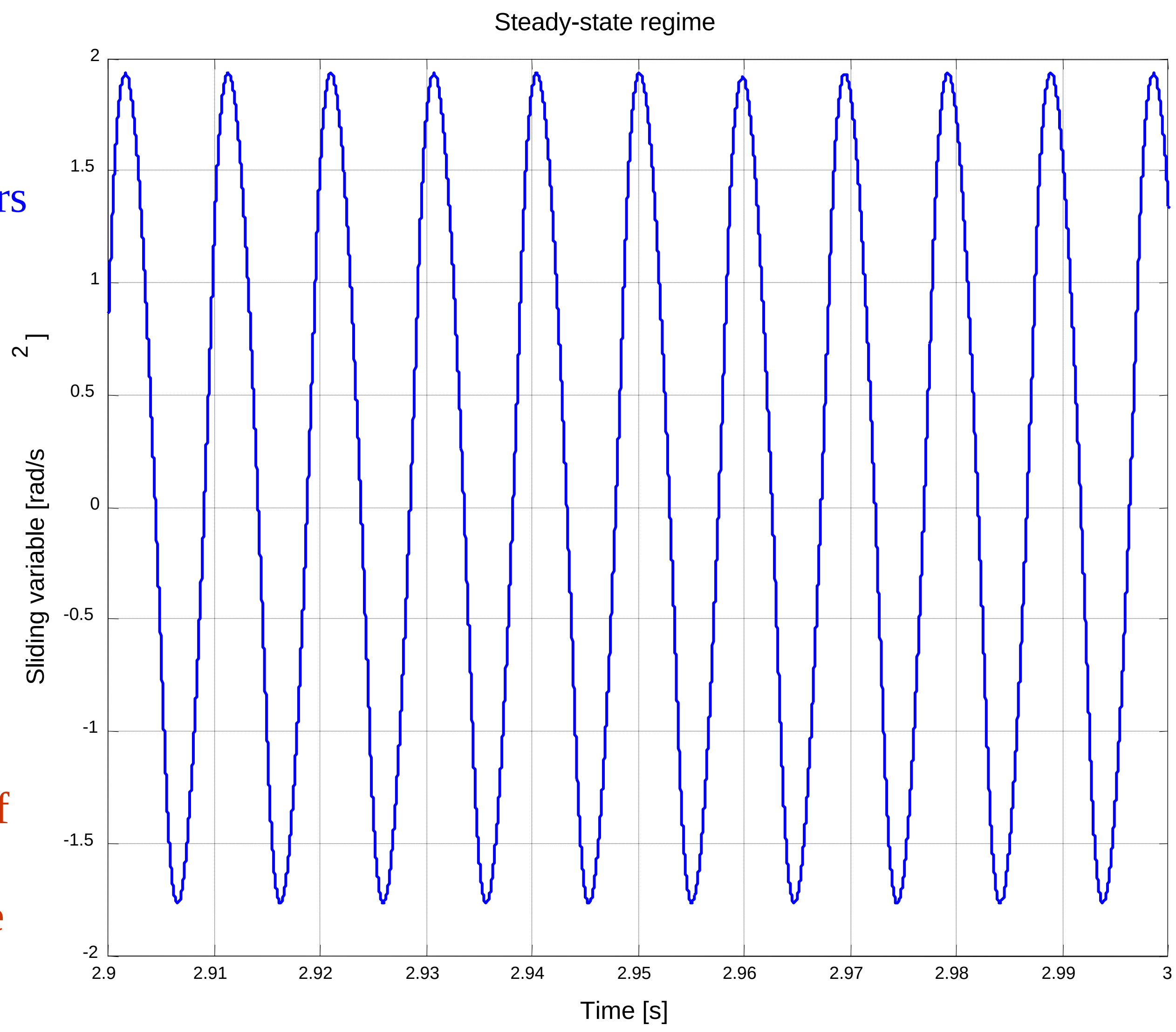
U_{ls} represents the magnitude of the oscillation of the sliding variable

DF Analysis & Sliding Modes

Nyquist plot parameters
ω_{lc}=662 rad/s
U_{lm}=1.1833 rad/s^2

Simulation results
T_{ls}=9.7 ms
ω_{lc}=646 rad/s
U_{lm}=1.8544 rad/s^2

Differences are due to:
- Numerical solution of the simulation
- Approximation of the DF approach

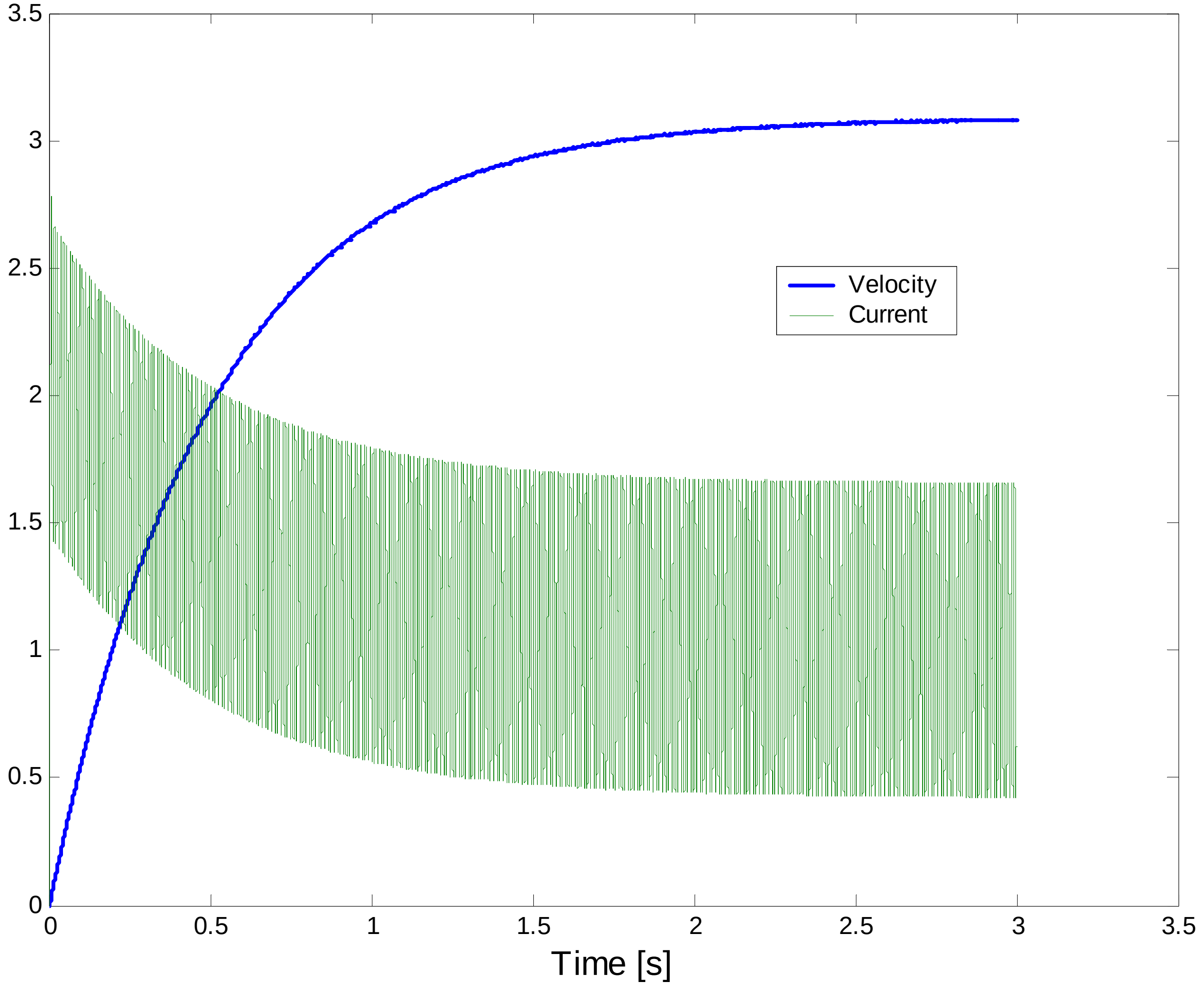


DF Analysis & Sliding Modes

The rotor velocity tends to its steady-state value as a first order system with time constant τ=0.5s

Rotor wound current presents large variations

An approximate sliding mode is established



DF Analysis & Sliding Modes

$$\begin{bmatrix} \dot{\omega} \\ \dot{i}_r \end{bmatrix} = \begin{bmatrix} \dfrac{B}{J} & \dfrac{k_t}{J} \\ -\dfrac{k_e}{L_r} & -\dfrac{R_r}{L_r} \end{bmatrix} \begin{bmatrix} \omega \\ i_r \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L_r} \end{bmatrix} v_r$$

Full system dynamics

$$\sigma = \begin{bmatrix} 2 + \dfrac{B}{J} & \dfrac{k_t}{J} \end{bmatrix} \begin{bmatrix} \omega \\ i_r \end{bmatrix}$$

$$\frac{k_t}{J} i_r = \sigma - \left(2 + \frac{B}{J} \right) \omega$$

Internal reduced-order dynamics $\dot{\omega} = -2\omega + s$

Input-output dynamics

$$\dot{\sigma} = \left(2 + \frac{B}{J} - \frac{R_r}{L_r} \right) \sigma + \left(\left(2 + \frac{B}{J} \right) \left(\frac{R_r}{L_r} - 2 \right) - \frac{k_e}{L_r} \frac{k_t}{J} \right) \omega + \frac{k_t}{J L_r} v_r$$

DF Analysis & Sliding Modes

Internal reduced-order dynamics $\dot{\omega} = -2\omega + \sigma$ $\Rightarrow$ $\Omega(s) = \dfrac{1}{s+2}\Sigma(s)$

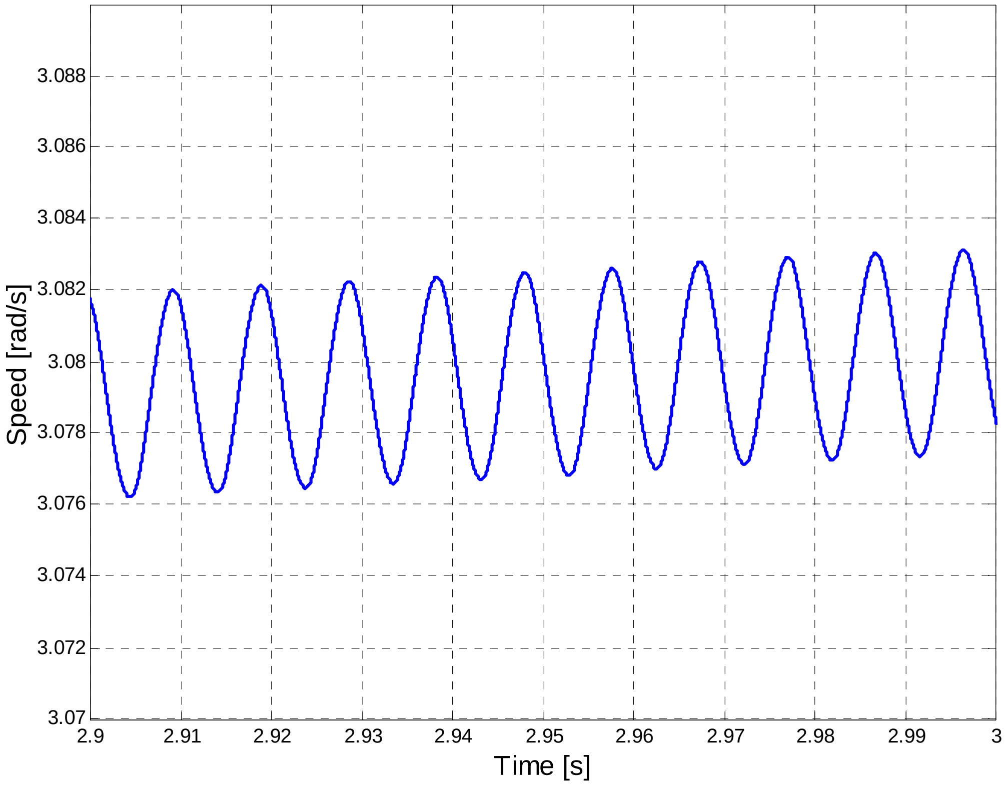
At steady-state

$$\sigma(t) = \sigma_0 + U\sin(\omega_{lc}t) \Rightarrow \omega(t) = \omega_0 + U\left|\frac{1}{j\omega_{lc}+2}\right|\sin\left(\omega_{lc}t + \arg\left(\frac{1}{j\omega_{lc}+2}\right)\right)$$

$$\left|\frac{1}{j\omega_{ls}+2}\right| \cong 0.0015$$

Nyquist plot parameters
ω_{lc}=662 rad/s
U_{ls}=1.1833 rad/s^2
$\Delta\omega_{ls}$=0.0018 rad/s

Simulation results
T_{ls}=9.7 ms
ω_{lc}=646 rad/s
U_{ls}=1.8544 rad/s^2
$\Delta\omega_{ls}$=0.0029 rad/s



DF Analysis & Sliding Modes

- ✓ The Describing Function approach to the analysis of the chattering phenomenon in sliding mode control systems can be used to have an estimate of the chattering parameters, i.e., frequency and magnitude
- ✓ The estimates are affected by an error which depends on the low-pass properties of the linear part of the plant
- ✓ The sliding variable must be considered as the output of the nonlinear feedback system
- ✓ The actual system output behaviour can be estimated by considering the reduced order dynamics
- ✓ In the presence of a constant reference value, the nonlinear function can be not symmetric, therefore an equivalent gain of the nonlinearity has to be considered to estimate the constant mean steady-state value of the output