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Fuzzy fractional initial value problem

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Abstract. We consider initial value problems for differential equations of fractional order with uncertainty and present the theory and some numerical methods to solve such type of problems under generalized differentiability conditions. The main tool is Banach fixed point theorem. Also we study the numerical approximation of the solutions of a fuzzy fractional initial value problem by using product trapezoidal and product rectangle formulas; the convergence of the numerical scheme is analyzed rigorously. Finally some numerical examples are provided to illustrate the applicability and usefulness of the obtained results.

Keywords: Fuzzy fractional initial value problem, Banach fixed point theorem, Product trapezoidal rule, Product rectangle rule

1. Introduction

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Recently a large amount of literature has been developed concerning the application of fuzzy fractional differential equations in other fields and related mathematical tools and techniques could be found in [2, 6, 9–11, 13, 14, 17].

In recent years, only few works have been reported on the development of theory and applications of fuzzy fractional differential equations. For instance, the existence and uniqueness results for fractional integral and differential equations with uncertainty and second-order fuzzy Volterra integrodifferential equations with kernel are proposed in [5, 16, 18]. Senol [19] proved the frequency boundary of fractional order systems with nonlinear uncertainties. Agarwal et al. [2] and Khastan et al. [10] have proposed the concept of solutions for fractional differential equations with uncertainty by considering Riemann-Liouville differentiability to solve fuzzy fractional differential equations. Further Allahviranloo et al. [4] investigated the explicit solutions of uncertain fractional differential equations under Riemann-Liouville H-differentiability using Mittag-Leffler functions and using a tau method with Jacobi polynomials for the solution of linear fractional differential equations based on the Caputo's H-differentiability [3].

The numerical methods have been developed for finding the solution of fuzzy fractional initial value problems by using the generalized Taylor's formula to $y(x_{j+1})$ about x_j and neglecting the second order term (involving $h^{2\beta}$) in [14]. But the generalized Taylor's formula depends on the lower limit of the integral in the definition of fractional derivative.

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Motivated by the above facts, in this paper study with the initial value problems for differential equations of fractional order with uncertainty and present the theory and some numerical methods to solve such type of problems under generalized differentiability conditions. The methods presented use the properties of fractional integral operators and a Banach fixed point theorem.

This paper deals with the existence and numerical solutions of fuzzy fractional initial value problems under generalized differentiability concept. In order to obtain the existence of solutions for fuzzy fractional initial value problem, we use Banach fixed point theorem. Further we show that the numerical solutions of fractional initial value problems obtained by product trapezoidal and product rectangle formulas are more accurate and agree well with the exact solutions. Finally the proposed methods are used to solve two examples.

2. Preliminaries

We denote by $\mathbb{R}_F$ the class of fuzzy subsets $u : \mathbb{R} \to [0, 1]$ satisfying the following properties:

(i) u is normal, that is, there exist $x_0 \in \mathbb{R}$ with $u(x_0) = 1$.

(ii) u is a convex fuzzy set, that is,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\},$$
$$\forall x, y \in \mathbb{R},\ \forall \lambda \in [0, 1].$$

(iii) u is upper semi-continuous on $\mathbb{R}$.

(iv) $\text{cl}\,\{x \in \mathbb{R} | u(x) > 0\}$ is compact where *cl* denotes the closure of a subset.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers. For $0 < \alpha \leq 1$, set $[u]^\alpha = \{s \in \mathbb{R} | u(s) \geq \alpha\}$ and $[u]^0 = \overline{\{s \in \mathbb{R} | u(s) > 0\}}$. Then the α- level set $[u]^\alpha$ is a nonempty compact interval for all $0 \leq \alpha \leq 1$ and any $u \in \mathbb{R}_F$. The notation $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches on u respectively.

For $u \in \mathbb{R}_F$, we define the length of u by: $len(u) = \overline{u} - \underline{u}$. For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall\ \alpha \in [0, 1]$ where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance $d : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$ defined by

$$d(u, v) = \sup_{\alpha \in [0,1]} max\{| \underline{u}^\alpha - \underline{v}^\alpha |, | \overline{u}^\alpha - \overline{v}^\alpha |\}.$$

Then it is easy to see that d is a metric in $\mathbb{R}_F$ and has the following properties:

(1) $d(u + w, v + w) = d(u, v)$,
(2) $d(\lambda u, \lambda v) = |\lambda| d(u, v)$,
(3) $d(u_1 + v_1, u_2 + v_2) \leq d(u_1, u_2) + d(v_1, v_2)$,
(4) $(\mathbb{R}_F, d)$ is a complete metric space,

for all $u, v, w \in \mathbb{R}_F$ and $\lambda \in R$.

Definition 2.1. Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$ then z is called the H-difference of x, y and it is denoted $x \ominus y$.
Throughout this paper, the sign "$\ominus$" always stands for the H-difference and we remark that $x \ominus y \neq x + (-1)y$ in general. Usually we denote $x + (-1)y$ by $x - y$. In the sequel, we fix $I = [a, b]$, for $a, b \in \mathbb{R}$.

Definition 2.2. Let $F : I \to \mathbb{R}_F$ and fix $t_0 \in (a, b)$. We say that F is (1)-differentiable at t_0, if there exists an element $F'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently near to 0 such that $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$
$$= F'(t_0) \text{ exist.}$$

We say that F is (2)-differentiable if for all $h > 0$ sufficiently near to 0 such that $F(t_0) \ominus F(t_0 + h)$, $F(t_0 - h) \ominus F(t_0)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h}$$
$$= F'(t_0) \text{ exist.}$$

If t_0 is the end point of I, then we consider the corresponding one-sided derivative.

Theorem 2.1. [21] *Let $F : [0, \infty) \to \mathbb{R}_F$. Assume that $\underline{F}^\alpha(x)$ and $\overline{F}^\alpha(x)$ are Riemann-integrable on $[a, b]$ for every $b \leq a$ and assume that there are two positive functions $\underline{M}^\alpha$, $\overline{M}^\alpha$ such that $\int_a^b \underline{F}^\alpha(x)dx \leq \underline{M}^\alpha$ and $\int_a^b \overline{F}^\alpha(x)dx \leq \overline{M}^\alpha$ for every $b \leq a$. Then $F(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore*

$$\left[\int_0^{\infty} F(x)dx\right]^{\alpha} = \left[\int_0^{\infty} \underline{F}^{\alpha}(x)dx, \int_0^{\infty} \overline{F}^{\alpha}(x)dx\right].$$

Theorem 2.2. *Let $F, G : I \to \mathbb{R}_F$ be integrable and $\lambda \in \mathbb{R}$. Then*

1. $\int_I (F(x) + G(x))dx = \int_I F(x)dx + \int_I G(x)dx$
2. $\int_I \lambda F(x)dx = \lambda \int_I F(x)dx$
3. $x \to d\,(F(x), G(x))$ *is integrable*
4. $d\left(\int_I F(x)dx, \int_I G(x)dx\right) \le \int_I d(F(x), G(x))dx.$

3. Fractional integral and fractional derivative of fuzzy number valued function

The space of all continuous fuzzy number valued functions on I, the space of all absolutely continuous fuzzy number valued functions on I and the space of all Lebesgue integrable fuzzy number valued functions on I are respectively denoted by $C^F(I)$, $(AC)^F(I)$ and $L^F(I)$. Throughout this paper, let $\beta \in (0, 1)$.

Definition 3.1. [10] Let $f \in L^F(I)$. The Riemann-Liouville fractional integral of order β of the fuzzy number valued function f is defined as follows:

$$J_a^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(\xi)}{(x-\xi)^{1-\beta}} d\xi, \quad x > a,$$

where $\Gamma(\beta)$ is the well-known Gamma function.

Theorem 3.1. [1, 6] *Let $f \in L^F(I)$. The Riemann-Liouville fractional integral of order β of the fuzzy number valued function f, based on its α-cut representation, can be expressed as*

$$\left[J_a^{\beta} f(x)\right]^{\alpha} = \left[J_a^{\beta} \underline{f}^{\alpha}(x), J_a^{\beta} \overline{f}^{\alpha}(x)\right], \quad x > a,$$

where

$$J_a^{\beta} \underline{f}^{\alpha}(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\underline{f}^{\alpha}(\xi)}{(x-\xi)^{1-\beta}} d\xi,$$

$$J_a^{\beta} \overline{f}^{\alpha}(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\overline{f}^{\alpha}(\xi)}{(x-\xi)^{1-\beta}} d\xi.$$

Definition 3.2. [12, 15, 20] If $f \in AC(I)$, then Riemann-Liouville fractional derivative of order β of the crisp function f exists almost everywhere on I and can be represented by

$$^{RL}{}_aD^{\beta} f(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x f(\xi)(x-\xi)^{-\beta} d\xi.$$

Note that Riemann-Liouville fractional derivative of order β of f is the first order derivative of the fractional integral $1 - \beta$ of f.

Definition 3.3. [12, 15, 20] If $f \in AC(I)$, then Caputo fractional derivative of order β of the crisp function f exists almost everywhere on I and can be represented by

$${}_a^C D^{\beta} f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x f'(\xi)(x-\xi)^{-\beta} d\xi.$$

Note that Caputo fractional derivative of order β of f is the fractional integral $1 - \beta$ of the first order derivative of f.

Definition 3.4. [14] Let $f \in (AC)^F(I)$ and

$$G(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x f(\xi)(x-\xi)^{-\beta} d\xi, \text{ for } x > a.$$

If the fuzzy number valued function G is (i) differentiable, then Riemann-Liouville fractional derivative of order β of the fuzzy number valued function f exists almost everywhere on I and can be represented by $^{RL}{}_aD_1^{\beta} f(x) = \frac{d}{dx} G(x)$. If the fuzzy number valued function G is (ii) differentiable, then Riemann-Liouville fractional derivative of order β of the fuzzy number valued function f exists almost everywhere on I and can be represented by $^{RL}{}_aD_2^{\beta} f(x) = \frac{d}{dx} G(x)$.

Definition 3.5. [14] Let $f \in (AC)^F(I)$. Then f is said to be the Caputo fractional differentiable, fuzzy number valued function of order β at $x \in (a, b)$, if

$${}_a^C D^{\beta} f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x f'(\xi)(x-\xi)^{-\beta} d\xi.$$

If the fuzzy number valued function f is (i) differentiable, then f is said to be Caputo differentiable in the first form and denoted by ${}_a^C D_1^{\beta} f(x)$. If $f(x)$ is (ii) differentiable, then f is said to be Caputo differentiable in the second form and denoted by ${}_a^C D_2^{\beta} f(x)$.

Theorem 3.2. [14] *Let $f(x) \in (AC)^F(I) \cap L^F(I)$. $[f(x)]^{\alpha} = [\underline{f}^{\alpha}(x), \overline{f}^{\alpha}(x)]$, for $x \in (a, b)$.*

(i) *If $f(x)$ is a Caputo fractional differentiable fuzzy number valued function in the first form, then*

$$[{}_a^C D_1^\beta f(x)]^\alpha = [{}_a^C D^\beta \underline{f}^\alpha(x), {}_a^C D^\beta \overline{f}^\alpha(x)].$$

(ii) *If $f(x)$ is a Caputo fractional differentiable fuzzy number valued function in the second form, then*

$$[{}_a^C D_2^\beta f(x)]^\alpha = [{}_a^C D^\beta \overline{f}^\alpha(x), {}_a^C D^\beta \underline{f}^\alpha(x)].$$

4. Fuzzy fractional initial value problem

In this section, existence of two solutions of fuzzy fractional initial value problem (FFIVP) under generalized differentiability by Banach fixed point theorem is presented. The predictor-corrector method for solving fuzzy fractional initial value problem is also be presented. The product rectangle rule is used as a prediction at each step and the product trapezoidal rule is used to make correction to obtain value at each step.
Consider the FFIVP

$$\left.\begin{array}{ll} {}_a^C D^\beta y(x) = f(x, y), & \beta \in (0, 1), \\ y(a) \quad = y_0, & x \in I, \;\; y_0, y \in \mathbb{R}_F. \end{array}\right\} \tag{1}$$

Definition 4.1. Let $y : I \to \mathbb{R}_F$ and $n \in \{1, 2\}$. We say that y is an n-solution for problem (1) on I, if ${}_a^C D_n^\beta y$ exists almost everywhere on I and ${}_a^C D_n^\beta y(x) = f(x, y(x))$, $y(a) = y_0$.

Lemma 4.1. *If the function f is continuous, then the FFIVP (1) is equivalent to the following integral equations*

$$y(x) = y_0 + \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t, y(t))}{(x - t)^{1-\beta}} dt, \;\; y(a) = y_0, \tag{2}$$

if $y(x)$ is Caputo differentiable of form ${}_a^C D_1^\beta y(x)$ and

$$\widehat{y}(x) = y_0 \ominus \frac{-1}{\Gamma(\beta)} \int_a^x \frac{f(t, \widehat{y}(t))}{(x - t)^{1-\beta}} dt, \;\; \widehat{y}(a) = y_0, \tag{3}$$

if $y(x)$ is Caputo differentiable of form ${}_a^C D_2^\beta y(x)$.

Theorem 4.2. *Let $R_0 = I \times \overline{B}(y_0, q)$, $q > 0$, $y_0 \in \mathbb{R}_F$, where $\overline{B}(y_0, q) = \{y \in \mathbb{R}_F : d(y, y_0) \le q\}$ denotes a closed ball in $\mathbb{R}_F$. Assume that $f(x, y(x))$ is a continuous function on $I \times \overline{B}(y_0, q)$ such that $d(\tilde{0}, f(x, y)) = \|f(x, y)\| \le M$ for all $(x, y) \in R_0$. Let f satisfy the Lipschitz condition $d(f(x, y), f(x, z)) \le Ld(y, z)$, for all $(x, y), (x, z) \in R_0$ with $L \le \frac{\Gamma(\beta+1)}{(b-a)^\beta}$ and $d(y, z) \le q$. Then the integral equations (2) and (3) have unique solution on [a,r] where $r = min \left[b, \left(\frac{q\Gamma(\beta+1)}{M}\right)^{\frac{1}{\beta}} + a\right]$.*

proof. Let $0 < \eta < r$ and define $U = \{y \in C_F[a, \eta];\; d(y, y_0) \le q\}$. (U, d) is a complete metric space. Here T is an operator on U and defined by

$$Ty(x) = y_0 + \frac{1}{\Gamma(\beta)} \int_a^x (x - t)^{\beta-1} f(t, y(t))dt.$$

To prove that T map U into U. Let $y \in U$ and $a \le x_1 < x_2 \le b$.

$$\begin{aligned}
& d(Ty(x_1), Ty(x_2)) \\
& = d\Big(\frac{1}{\Gamma(\beta)} \int_a^{x_1} (x_1 - t)^{\beta-1} f(t, y(t))dt, \\
& \qquad \frac{1}{\Gamma(\beta)} \int_a^{x_2} (x_2 - t)^{\beta-1} f(t, y(t))dt\Big) \\
& \le \frac{1}{\Gamma(\beta)} d\Big(\int_a^{x_1} (x_1 - t)^{\beta-1} f(t, y(t))dt, \\
& \qquad \int_a^{x_1} (x_2 - t)^{\beta-1} f(t, y(t))dt\Big) \\
& + \frac{1}{\Gamma(\beta)} d\left(\tilde{0}, \int_{x_1}^{x_2} (x_2 - t)^{\beta-1} f(t, y(t))dt\right) \\
& \le \frac{1}{\Gamma(\beta)} \int_a^{x_1} d((x_1 - t)^{\beta-1} f(t, y(t)), \\
& \quad (x_2 - t)^{\beta-1} f(t, y(t)))dt \\
& + \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_2} (x_2 - t)^{\beta-1} d(\tilde{0}, f(t, y(t)))dt \\
& \le \frac{1}{\Gamma(\beta)} \int_a^{x_1} |(x_1 - t)^{\beta-1} \\
& - (x_2 - t)^{\beta-1}| d(f(t, y(t)), \tilde{0})dt \\
& + \frac{M}{\Gamma(\beta)} \int_{x_1}^{x_2} (x_2 - t)^{\beta-1} dt \\
& \le \frac{M}{\Gamma(\beta)} \Big(\int_a^{x_1} ((x_1 - t)^{\beta-1} \\
& - (x_2 - t)^{\beta-1})dt + \frac{2}{\beta}(x_2 - x_1)^\beta\Big) \\
& \le \frac{2M}{\Gamma(\beta + 1)} (x_2 - x_1)^\beta.
\end{aligned}$$

Thus Ty is continuous. Also

$$\begin{aligned} d(Ty(x), y_0) &= \frac{1}{\Gamma(\beta)} d\left(\int_a^x (x-t)^{\beta-1} f(t, y(t))dt, \tilde{0}\right) \\ &\le \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} d\left(f(t, y(t)), \tilde{0}\right) dt \\ &\le \frac{M}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} dt \\ &\le \frac{M}{\Gamma(\beta+1)} (b-a)^{\beta} \le q. \end{aligned}$$

Thus T maps U into U. Next we show that T is a contraction. In fact, for $y_1, y_2 \in C_F[a, \eta]$,

$$\begin{aligned} & d(Ty_1(x), Ty_2(x)) \\ &= \frac{1}{\Gamma(\beta)} d\left(\int_a^x (x-t)^{\beta-1} f(t, y_1(t))dt, \right. \\ &\qquad \left. \int_a^x (x-t)^{\beta-1} f(t, y_2(t))dt\right) \\ &\le \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} d\left(f(t, y_1(t)), f(t, y_2(t))\right) dt \\ &\le L \frac{d(y_1, y_2)}{\Gamma(\beta+1)} (x-a)^{\beta}. \end{aligned}$$

By the Banach contraction principle, T has a unique fixed point which is a solution of the integral equation (2). Here T_1 is an operator on U and defined by

$$T_1 y(x) = y_0 \ominus \frac{-1}{\Gamma(\beta)} \int_a^x \frac{f(t, \widehat{y}(t))}{(x-t)^{1-\beta}} dt.$$

By the invariance of translation of distance d and using $d(\lambda u, \lambda v) = |\lambda| d(u, v)$, T_1 satisfies the conditions for Banach contraction principle; so T_1 has a unique fixed point which is a solution of the integral equation (3). □

Theorem 4.3. *We consider the FFIVP (1) where $f : I \times \mathbb{R}_F \to \mathbb{R}_F$ is such that*

(i) $[f(x, y)]^{\alpha} = [\underline{f}^{\alpha}(x, \underline{y}, \overline{y}), \overline{f}^{\alpha}(x, \underline{y}, \overline{y})]$;

(ii) $\{\underline{f}^{\alpha}; \alpha \in [0, 1]\}$ *and* $\{\overline{f}^{\alpha}; \alpha \in [0, 1]\}$ *are equicontinuous (that is, for any $\epsilon > 0$ and any $(x, y, z) \in I \times \mathbb{R}^2$, we have $|\underline{f}^{\alpha}(x, y, z) - \underline{f}^{\alpha}(x_1, y_1, z_1)| < \epsilon$, $|\overline{f}^{\alpha}(x, y, z) - \overline{f}^{\alpha}(x_1, y_1, z_1)| < \epsilon, \forall \alpha \in [0, 1]$, whenever $\|(x_1, y_1, z_1) - (x, y, z)\| < \delta$) and uniformly bounded on any bounded set;*

(iii) *There exists $L > 0$ such that*

$$\begin{aligned} & |\underline{f}^{\alpha}(x, y_1, z_1) - \underline{f}^{\alpha}(x, y_2, z_2)| \\ &\quad \le L \max\{|y_1 - y_2|, |z_1 - z_2|\} \ \forall \alpha \in [0, 1], \\ & |\overline{f}^{\alpha}(x, y_1, z_1) - \overline{f}^{\alpha}(x, y_2, z_2)| \\ &\quad \le L \max\{|y_1 - y_2|, |z_1 - z_2|\} \ \forall \alpha \in [0, 1]. \end{aligned}$$

Then the FFIVP (1) is equivalent to the following systems of fractional ordinary differential equations

$$\left.\begin{aligned} {}_a^C D_1^{\beta} \underline{y}^{\alpha}(x) &= \underline{[f(x, y)]}^{\alpha} = F(x, \underline{y}^{\alpha}, \overline{y}^{\alpha}), \\ {}_a^C D_1^{\beta} \overline{y}^{\alpha}(x) &= \overline{[f(x, y)]}^{\alpha} = G(x, \underline{y}^{\alpha}, \overline{y}^{\alpha}), \\ \underline{y}^{\alpha}(a) &= \underline{y}_0^{\alpha}, \ \ \overline{y}^{\alpha}(a) = \overline{y}_0^{\alpha}, \ \ x \in I. \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} {}_a^C D_2^{\beta} \underline{y}^{\alpha}(x) &= \overline{[f(x, y)]}^{\alpha} = G(x, \underline{y}^{\alpha}, \overline{y}^{\alpha}), \\ {}_a^C D_2^{\beta} \overline{y}^{\alpha}(x) &= \underline{[f(x, y)]}^{\alpha} = F(x, \underline{y}^{\alpha}, \overline{y}^{\alpha}), \\ \underline{y}^{\alpha}(a) &= \underline{y}_0^{\alpha}, \ \ \overline{y}^{\alpha}(a) = \overline{y}_0^{\alpha}, \ \ x \in I. \end{aligned}\right\} \tag{5}$$

when y is Caputo differentiable in the first form and second form respectively.

proof. The equicontinuity of $\underline{f}^{\alpha}, \overline{f}^{\alpha}$ implies the continuity of the function f. Further the Lipschitz property in (iii) ensures that f satisfies a Lipschitz property in the following form

$$\begin{aligned} & \sup_{\alpha \in [0,1]} \max\{|\underline{f}^{\alpha}(x, \underline{y}^{\alpha}, \overline{y}^{\alpha}) - \underline{f}^{\alpha}(x, \underline{z}^{\alpha}, \overline{z}^{\alpha})|, \\ &\qquad |\overline{f}^{\alpha}(x, \underline{y}^{\alpha}, \overline{y}^{\alpha}) - \overline{f}^{\alpha}(x, \underline{z}^{\alpha}, \overline{z}^{\alpha})|\} \\ &\le L \sup_{\alpha \in [0,1]} \max\{|\underline{y}^{\alpha} - \underline{z}^{\alpha}|, |^{\alpha} - \overline{z}^{\alpha}|\}; \end{aligned}$$

that is,

$$d(f(x, y), f(x, z)) \le L d(y, z). \tag{6}$$

By the continuity of f, from the last Lipschitz condition, boundedness condition (ii) and Theorem 4.2, it follows that (1) has two solutions $y, \ \hat{y}$ corresponding to Caputo differentiability in the first form and second form respectively. By Theorem 3.1, the functions $\underline{y}^{\alpha}(x)$ and $\overline{y}^{\alpha}(x)$ are differentiable in the first and $\underline{\hat{y}}^{\alpha}(x)$ and $\overline{\hat{y}}^{\alpha}(x)$ are differentiable in the second forms and as conclusion $(\underline{y}^{\alpha}(x), \overline{y}^{\alpha}(x))$ is a solution of (4) and $(\underline{\hat{y}}^{\alpha}(x), \overline{\hat{y}}^{\alpha}(x))$ is a solution of (5).

Conversely, we suppose that we have a solution $\underline{y}^{\alpha}(x), \overline{y}^{\alpha}(x)$ with $\alpha \in [0, 1]$ of the system (4) and a solution $\underline{\hat{y}}^{\alpha}(x), \overline{\hat{y}}^{\alpha}(x)$ with $\alpha \in [0, 1]$ fixed of the system (5). Also the Lipschitz condition (6) implies the existence of two solutions z, $\hat{z}$ of the FFIVP (1) for Caputo-differentiable in the first and second forms respectively. Since z is Caputo -differentiable of form ${}_a^C D_1^\beta z(x)$, $(\underline{z}^\alpha, \overline{z}^\alpha)$ the end points of z^α is a solution of (4). Since the solution of (4) is unique, we have $z^\alpha = [\underline{z}^\alpha, \overline{z}^\alpha] = [\underline{y}^\alpha, \overline{y}^\alpha]$, that is, the problem (1) and (4) are equivalent when y is Caputo -differentiable of form ${}_a^C D_1^\beta y(x)$. Similarly when $\hat{y}$ is Caputo-differentiable of form ${}_a^C D_2^\beta y(x)$, we have $\hat{z}^\alpha = [\underline{\hat{z}}^\alpha, \overline{\hat{z}}^\alpha] = [\underline{\hat{y}}^\alpha, \overline{\hat{y}}^\alpha]$, that is, the problem (1) and (5) are equivalent. $\square$

Numerical method for (1) is the same for Caputo-differentiable of the two forms. We assume that y is Caputo -differentiable of form ${}_a^C D_1^\beta y(x)$.

The initial value problem (4) is equivalent to the following integral equations:

$$\left.\begin{aligned} \underline{y}^\alpha(x) &= J_a^\beta F(x, \underline{y}^\alpha, \overline{y}^\alpha) + \underline{y}^\alpha(a), \\ \overline{y}^\alpha(x) &= J_a^\beta G(x, \underline{y}^\alpha, \overline{y}^\alpha) + \overline{y}^\alpha(a), \\ \underline{y}^\alpha(a) &= \underline{y}_0^\alpha, \quad \overline{y}^\alpha(a) = \overline{y}_0^\alpha, \quad x \in I. \end{aligned}\right\} \tag{7}$$

We introduce a uniform grid of points x_j with $x_j = jh$, $j = 0, 1, 2, ...m$, where m is a positive integer, $h = \frac{b-a}{m}$ is the mesh-width in I. Exact and approximate solutions of (7) at the grid point x_j are denoted by $(\underline{y}^\alpha(x_j), \overline{y}^\alpha(x_j))$ and $(\underline{y}_j^\alpha, \overline{y}_j^\alpha)$ respectively. Using the product trapezoidal rule [7, 8] for the integral $J_a^\beta g(x)$ $(0 < \beta < 1)$, we obtain

$$J_a^\beta g(x_j) = \frac{1}{\Gamma(\beta)} \sum_{i=0}^{j} a_{i,j} g(x_i), \tag{8}$$

where

$$a_{i,j} = \frac{h^\beta}{\beta(\beta+1)} \begin{cases} (j-1)^{\beta+1} - j^\beta(j-1-\beta), i = 0, \\ (j-i+1)^{\beta+1} - 2(j-i)^{\beta+1} \\ +(j-i-1)^{\beta+1}, \ 1 \le i \le j-1, \\ 1, \ \ i = j. \end{cases}$$

Then (8) and Theorem 3.1 give the corrector formulas for (7) which is

$$\left.\begin{aligned} \underline{y}_j^\alpha &= \tfrac{1}{\Gamma(\beta)} \Big[\sum_{i=0}^{j-1} a_{i,j} F(x_i, \underline{y}_i^\alpha, \overline{y}_i^\alpha) \\ &\quad + \tfrac{h^\beta}{\beta(\beta+1)} F(x_j, (\underline{y}_j^\alpha)^p, (\overline{y}_j^\alpha)^p) \Big] + \underline{y}^\alpha(a) \\ \overline{y}_j^\alpha &= \tfrac{1}{\Gamma(\beta)} \Big[\sum_{i=0}^{j-1} a_{i,j} G(x_i, \underline{y}_i^\alpha, \overline{y}_i^\alpha) \\ &\quad + \tfrac{h^\beta}{\beta(\beta+1)} G(x_j, (\underline{y}_j^\alpha)^p, (\overline{y}_j^\alpha)^p) \Big] + \overline{y}^\alpha(a). \end{aligned}\right\} \tag{9}$$

The remaining problem is the determination of the predictor formula required to calculate $(\underline{y}_j^\alpha)^p$ and $(\overline{y}_j^\alpha)^p$. We replace the integral $J_a^\beta g(x)$ $(0 < \beta < 1)$ by the product rectangle rule [7, 8]

$$J_a^\beta g(x_j) = \frac{1}{\Gamma(\beta)} \sum_{i=0}^{j-1} b_{i,j} g(x_i),$$

where

$$b_{i,j} = \frac{h^\beta}{\beta}((j-i)^\beta - (j-i-1)^\beta).$$

Thus the predictors $(\underline{y}_j^\alpha)^p$ and $(\overline{y}_j^\alpha)^p$ of (9) are determined by

$$\left.\begin{aligned} (\underline{y}_j^\alpha)^p &= \underline{y}^\alpha(a) + \tfrac{1}{\Gamma(\beta)} \sum_{i=0}^{j-1} b_{i,j} F(x_i, \underline{y}_i^\alpha, \overline{y}_i^\alpha), \\ (\overline{y}_j^\alpha)^p &= \overline{y}^\alpha(a) + \tfrac{1}{\Gamma(\beta)} \sum_{i=0}^{j-1} b_{i,j} G(x_i, \underline{y}_i^\alpha, \overline{y}_i^\alpha). \end{aligned}\right\} \tag{10}$$

4.1. Convergence

Theorem 4.4. [8] *Let $z \in C^1[a, b]$. Then*

$$\left| \int_0^{x_j} (x_j - \xi)^{\beta-1} z(\xi) d\xi - \sum_{i=0}^{j-1} b_{i,j} z(x_i) \right| \le \frac{1}{\beta} \|z'\|_\infty x_j^\beta h.$$

Theorem 4.5. [8] *If $z \in C^2[a, b]$, then there exists a constant C_β^{Tr} depending only on β such that*

$$\left| \int_0^{x_j} (x_j - \xi)^{\beta-1} z(\xi) d\xi - \sum_{i=0}^{j} a_{i,j} z(x_i) \right| \le C_\beta^{Tr} \|z'\|_\infty x_j^\beta h^2.$$

Theorem 4.6. *Assume that f in FFIVP (1) satisfies the assumptions (i)-(iii) in Theorem 4.3 and assume that the solution y of FFIVP (1) is such that*

$$\left|\int_a^{x_j}(x_j-\xi)^{\beta-1}\,{}_a^C D^\beta \underline{y}^\alpha(\xi)d\xi-\sum_{i=0}^{j-1} b_{i,j}\,{}_a^C D^\beta \underline{y}^\alpha(x_i)\right|$$
$$\leq C_1 x_j^\beta h, \tag{11}$$
$$\left|\int_a^{x_j}(x_j-\xi)^{\beta-1}\,{}_a^C D^\beta \overline{y}^\alpha(\xi)d\xi-\sum_{i=0}^{j-1} b_{i,j}\,{}_a^C D^\beta \overline{y}^\alpha(x_i)\right|$$
$$\leq C_1 x_j^\beta h,$$

and

$$\left|\int_a^{x_j}(x_j-\xi)^{\beta-1}\,{}_a^C D^\beta \underline{y}^\alpha(\xi)d\xi-\sum_{i=0}^{j} a_{i,j}\,{}_a^C D^\beta \underline{y}^\alpha(x_i)\right|$$
$$\leq C_2 x_j^\beta h^2, \tag{12}$$
$$\left|\int_a^{x_j}(x_j-\xi)^{\beta-1}\,{}_a^C D^\beta \overline{y}^\alpha(\xi)d\xi-\sum_{i=0}^{j} a_{i,j}\,{}_a^C D^\beta \overline{y}^\alpha(x_i)\right|$$
$$\leq C_2 x_j^\beta h^2.$$

with some $\beta \geq 0$. Then we have

$$\max_{0\leq l\leq m}|\underline{y}^\alpha(x_l)-\underline{y}_l^\alpha| = O(h^q),$$
$$\max_{0\leq l\leq m}|\overline{y}^\alpha(x_l)-\overline{y}_l^\alpha| = O(h^q),\ q=min(1+\beta,2)$$

proof. We show for sufficiently small h, that

$$|\underline{y}^\alpha(x_l)-\underline{y}_l^\alpha| \leq A_1 h^q, \tag{13}$$
$$|\overline{y}^\alpha(x_l)-\overline{y}_l^\alpha| \leq A_2 h^q,$$

for all $l \in \{0, 1, 2, \dots m\}$ where A_1, A_2 are constants. The proof is based on mathematical induction. In view of given initial condition, it is true for ($l = 0$). Assume that it is true for some $l = j,\ \ j > 0$. We have to prove that the inequality is true for $l = j+1$. To do this, we find the error of the predictor $(y_{j+1}^\alpha)^p$.

$$|\underline{y}^\alpha(x_{j+1})-(\underline{y}_{j+1}^\alpha)^p|$$
$$=\frac{1}{\Gamma(\beta)}\left|\int_a^{x_{j+1}}(x_{j+1}-\xi)^{\beta-1}\ F(\xi,\underline{y}^\alpha(\xi),\overline{y}^\alpha(\xi))d\xi\right.$$
$$\left.-\sum_{i=0}^{j} b_{i,j+1}F(x_i,\underline{y}_i^\alpha,\overline{y}_i^\alpha)\right|$$
$$\leq\frac{1}{\Gamma(\beta)}\left|\int_a^{x_{j+1}}(x_{j+1}-\xi)^{\beta-1}\,{}_a^C D^\beta\underline{y}^\alpha(\xi)d\xi\right.$$
$$\left.-\sum_{i=0}^{j} b_{i,j+1}\,{}_a^C D^\beta\underline{y}^\alpha(x_i)\right|$$
$$+\sum_{i=0}^{j} b_{i,j+1}\left|F(x_i,\underline{y}^\alpha(x_i),\overline{y}^\alpha(x_i))-F(x_i,\underline{y}_i^\alpha,\overline{y}_i^\alpha)\right|$$
$$\leq\frac{C_1x_{j+1}^\beta}{\Gamma(\beta)}h+\frac{1}{\Gamma(\beta)}\sum_{i=0}^{j} b_{i,j+1}\max[|\underline{y}^\alpha(x_i)-\underline{y}_i^\alpha|,$$
$$|\overline{y}^\alpha(x_i)-\overline{y}_i^\alpha|]$$
$$\leq\frac{C_1x_{j+1}^\beta}{\Gamma(\beta)}h+\frac{LA}{\Gamma(\beta)}h^q\sum_{i=0}^{j} b_{i,j+1}$$
$$\leq\frac{C_1x_{j+1}^\beta}{\Gamma(\beta)}h+\frac{LA}{\Gamma(\beta+1)}b^\beta h^q.$$

Let $A = \max\{A_1, A_2\}$; we used the Lipschitz condition of F, the assumption on the error of the rectangle formula and the fact that, by construction of the quadrature formula underlying the predictor, $b_{i,j} \geq 0$ for all i and j and $\sum_{i=0}^{j} b_{i,j+1} = \int_a^{x_{j+1}}(x_{j+1}-\xi)^{\beta-1}d\xi \leq \frac{1}{\beta}b^\beta$. Similarly

$$|\overline{y}^\alpha(x_{j+1})-(\overline{y}_{j+1}^\alpha)^p| \leq \frac{C_1x_{j+1}^\beta}{\Gamma(\beta)}h+\frac{LA}{\Gamma(\beta+1)}b^\beta h^q.$$

The error analysis of the corrector formula gives

$$|\underline{y}^\alpha(x_{j+1})-\underline{y}_{j+1}^\alpha|$$
$$=\frac{1}{\Gamma(\beta)}\left|\int_a^{x_{j+1}}(x_{j+1}-\xi)^{\beta-1}\ F(\xi,\underline{y}^\alpha(\xi),\overline{y}^\alpha(\xi))d\xi\right.$$
$$-\sum_{i=0}^{j} a_{i,j+1}F(x_i,\underline{y}_i^\alpha,\overline{y}_i^\alpha)$$
$$\left.-F(x_{j+1},(\underline{y}_{j+1}^\alpha)^p,(\overline{y}_{j+1}^\alpha)^p)\right|$$
$$\leq\frac{1}{\Gamma(\beta)}\left|\int_a^{x_{j+1}}(x_{j+1}-\xi)^{\beta-1}\,{}_a^C D^\beta\underline{y}^\alpha(\xi)d\xi\right.$$
$$\left.-\sum_{i=0}^{j+1} a_{i,j+1}\,{}_a^C D^\beta\underline{y}^\alpha(x_i)\right|$$

Table 1
Numerical values of exact solutions $\underline{y}^{\alpha}(x_j)(1 \leq j \leq 10)$

x $\vert\alpha$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.
0.1	0.743382	0.81772	0.892058	0.966396	1.04073	1.11507	1.18941	1.26375	1.33809	1.41243	1.48676
0.2	0.899509	0.989459	1.07941	1.16936	1.25931	1.34926	1.43921	1.52916	1.61912	1.70907	1.79902
0.3	1.05385	1.15923	1.26462	1.37	1.47539	1.58077	1.68616	1.79154	1.89693	2.00231	2.1077
0.4	1.21502	1.33652	1.45803	1.57953	1.70103	1.82253	1.94403	2.06554	2.18704	2.30854	2.43004
0.5	1.38714	1.52586	1.66457	1.80329	1.942	2.08071	2.21943	2.35814	2.49686	2.63557	2.77429
0.6	1.57311	1.73042	1.88773	2.04504	2.20235	2.35966	2.51697	2.67428	2.83159	2.9889	3.14621
0.7	1.7754	1.95294	2.13048	2.30802	2.48556	2.6631	2.84064	3.01818	3.19572	3.37326	3.5508
0.8	1.99642	2.19606	2.3957	2.59534	2.79499	2.99463	3.19427	3.39391	3.59355	3.79319	3.99284
0.9	2.23859	2.46245	2.68631	2.91017	3.13403	3.35789	3.58175	3.80561	4.02947	4.25333	4.47718
1	2.50449	2.75494	3.00539	3.25584	3.50629	3.75674	4.00718	4.25763	4.50808	4.75853	5.00898

Table 2
Numerical values of exact solutions $\overline{y}^{\alpha}(x_j)(1 \leq j \leq 10)$

x $\vert\alpha$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1.5	1.45	1.4	1.35	1.3	1.25	1.2	1.15	1.1	1.05	1.
0.1	2.23015	2.15581	2.08147	2.00713	1.93279	1.85845	1.78412	1.70978	1.63544	1.5611	1.48676
0.2	2.69853	2.60858	2.51862	2.42867	2.33872	2.24877	2.15882	2.06887	1.97892	1.88897	1.79902
0.3	3.16155	3.05616	2.95078	2.84539	2.74001	2.63462	2.52924	2.42385	2.31847	2.21308	2.1077
0.4	3.64506	3.52356	3.40206	3.28056	3.15906	3.03755	2.91605	2.79455	2.67305	2.55155	2.43004
0.5	4.16143	4.02271	3.884	3.74529	3.60657	3.46786	3.32914	3.19043	3.05171	2.913	2.77429
0.6	4.71932	4.56201	4.4047	4.24739	4.09008	3.93277	3.77546	3.61814	3.46083	3.30352	3.14621
0.7	5.3262	5.14866	4.97112	4.79358	4.61604	4.4385	4.26096	4.08342	3.90588	3.72834	3.5508
0.8	5.98925	5.78961	5.58997	5.39033	5.19069	4.99104	4.7914	4.59176	4.39212	4.19248	3.99284
0.9	6.71578	6.49192	6.26806	6.0442	5.82034	5.59648	5.37262	5.14876	4.9249	4.70104	4.47718
1	7.51347	7.26302	7.01257	6.76212	6.51167	6.26123	6.01078	5.76033	5.50988	5.25943	5.00898

$$
\begin{aligned}
&+\sum_{i=0}^{j} a_{i,j+1}\Big|F(x_i,\underline{y}^{\alpha}(x_i),\overline{y}^{\alpha}(x_i))\\
&-F(x_i,\underline{y}_i^{\alpha},\overline{y}_i^{\alpha})\Big|\\
&+\Big|F(x_{j+1},\underline{y}^{\alpha}(x_{j+1}),\overline{y}^{\alpha}(x_{j+1}))\\
&\quad - F(x_{j+1},(\underline{y}_{j+1}^{\alpha})^p,(\overline{y}_{j+1}^{\alpha})^p)\Big|\\
\leq\ & \frac{C_2x_{j+1}^{\beta}}{\Gamma(\beta)}h^2+\frac{L}{\Gamma(\beta)}\sum_{i=0}^{j}a_{i,j+1}\max[|\underline{y}^{\alpha}(x_i)-\underline{y}_i^{\alpha}|,\\
&|\overline{y}^{\alpha}(x_i)-\overline{y}_i^{\alpha}|]+\frac{L}{\Gamma(\beta)}\max[|\underline{y}^{\alpha}(x_{j+1})-(\underline{y}_{j+1}^{\alpha})^p|,\\
&|\overline{y}^{\alpha}(x_{j+1})-(\overline{y}_{j+1}^{\alpha})^p|]\\
\leq\ & \frac{C_2x_{j+1}^{\beta}}{\Gamma(\beta+1)}h^2+\frac{LA}{\Gamma(\beta)}h^q\sum_{i=0}^{j}a_{i,j+1}\\
&+\frac{L}{\Gamma(\beta+2)}\left(\frac{C_1x_{j+1}^{\beta}}{\Gamma(\beta)}h+\frac{L(A)}{\Gamma(\beta+1)}b^{\beta}h^q\right)\\
\leq\ & \frac{C_2x_{j+1}^{\beta}}{\Gamma(\beta)}h^2+\frac{LA}{\Gamma(\beta+1)}b^{\beta}h^q+\\
&\frac{L}{\Gamma(\beta+2)}\left(\frac{C_1x_{j+1}^{\beta}}{\Gamma(\beta)}h+\frac{LA}{\Gamma(\beta+1)}b^{\beta}h^q\right).
\end{aligned}
$$

□

5. Numerical Examples

Example 1. We consider the linear fuzzy fractional initial value problem of the form

$$
\left.\begin{aligned}
{}_0^C D^{\beta}y(x) &= y(x),\ 0<\beta\leq 1,\quad x\in[0,1],\\
y(0) &= (0.5\alpha+0.5,-0.5\alpha+1.5),
\end{aligned}\right\}\tag{14}
$$

where $y(0)\in\mathbb{R}_F$.

Case(i) ${}_0^C D^{\beta}y$ is differentiable in first form. That is, $[{}_0^C D_1^{\beta}y(x)]^{\alpha}=[{}_0^C D^{\beta}\underline{y}^{\alpha}(x),{}_0^C D^{\beta}\overline{y}^{\alpha}(x)]$ and satisfies the 1-system associated with (14). Since in the crisp case, the solution of the following equation is $u_0E_{\beta}(x^{\beta})$,

$$
\begin{aligned}
&{}_0^C D^{\beta}u(x)=u(x),\ x\in[0,1]\\
&\quad u(0)=u_0,\ \text{where } u \text{ is a crisp function,}
\end{aligned}
$$

Table 3
Numerical values of the approximate solutions $\underline{y}_j^{\alpha}(1 \leq j \leq 10)$

x |α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.
0.1	0.720854	0.792939	0.865024	0.93711	1.0092	1.08128	1.15337	1.22545	1.29754	1.36962	1.44171
0.2	0.874604	0.962065	1.04953	1.13699	1.22445	1.31191	1.39937	1.48683	1.57429	1.66175	1.74921
0.3	1.02425	1.12667	1.2291	1.33152	1.43395	1.53637	1.6388	1.74122	1.84365	1.94607	2.0485
0.4	1.17962	1.29758	1.41554	1.53351	1.65147	1.76943	1.88739	2.00535	2.12332	2.24128	2.35924
0.5	1.34493	1.47942	1.61391	1.7484	1.8829	2.01739	2.15188	2.28637	2.42087	2.55536	2.68985
0.6	1.52298	1.67528	1.82758	1.97988	2.13218	2.28448	2.43678	2.58907	2.74137	2.89367	3.04597
0.7	1.71616	1.88778	2.0594	2.23101	2.40263	2.57425	2.74586	2.91748	3.0891	3.26071	3.43233
0.8	1.9267	2.11937	2.31204	2.50471	2.69738	2.89005	3.08272	3.27539	3.46806	3.66073	3.8534
0.9	2.15686	2.37254	2.58823	2.80391	3.0196	3.23528	3.45097	3.66666	3.88234	4.09803	4.31371
1	2.40899	2.64989	2.89078	3.13168	3.37258	3.61348	3.85438	4.09528	4.33618	4.57708	4.81797

Table 4
Numerical values of the approximate solutions $\overline{y}_j^{\alpha}(1 \leq j \leq 10)$

x |α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1.5	1.45	1.4	1.35	1.3	1.25	1.2	1.15	1.1	1.05	1.
0.1	2.16256	2.09048	2.01839	1.94631	1.87422	1.80213	1.73005	1.65796	1.58588	1.51379	1.44171
0.2	2.62381	2.53635	2.44889	2.36143	2.27397	2.18651	2.09905	2.01159	1.92413	1.83667	1.74921
0.3	3.07275	2.97032	2.8679	2.76547	2.66305	2.56062	2.4582	2.35577	2.25335	2.15092	2.0485
0.4	3.53886	3.4209	3.30294	3.18497	3.06701	2.94905	2.83109	2.71313	2.59516	2.4772	2.35924
0.5	4.03478	3.90028	3.76579	3.6313	3.49681	3.36231	3.22782	3.09333	2.95884	2.82434	2.68985
0.6	4.56895	4.41666	4.26436	4.11206	3.95976	3.80746	3.65516	3.50286	3.35057	3.19827	3.04597
0.7	5.14849	4.97688	4.80526	4.63364	4.46203	4.29041	4.11879	3.94718	3.77556	3.60395	3.43233
0.8	5.78011	5.58744	5.39477	5.2021	5.00943	4.81676	4.62409	4.43142	4.23875	4.04608	3.8534
0.9	6.47057	6.25488	6.0392	5.82351	5.60783	5.39214	5.17645	4.96077	4.74508	4.5294	4.31371
1	7.22696	6.98606	6.74516	6.50426	6.26337	6.02247	5.78157	5.54067	5.29977	5.05887	4.81797

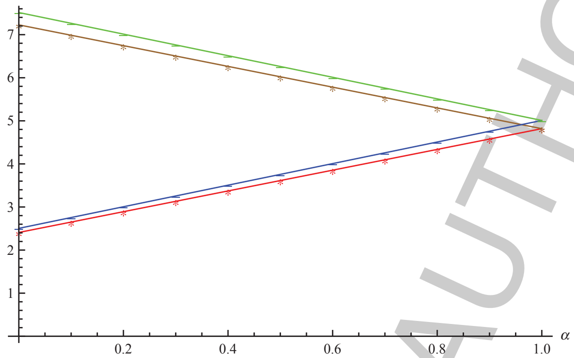


Fig. 1. Exact (-)(Blue, Green) and approximate values of (*)(Red, Brown) of lower and upper values of y at $x = 1$ when $m = 10$.

Table 5
The maximum errors and convergence orders of numerical scheme when $\beta = 0.5$

h	$\underline{E}_{\infty}$(h)	Order1	$\overline{E}_{\infty}$(h)	Order2
$\frac{1}{80}$	0.00980997	1.47656	0.014715	1.47656
$\frac{1}{160}$	0.00352515	1.48465	0.00528772	1.48465
$\frac{1}{320}$	0.00125966	1.48963	0.00188949	1.48963
$\frac{1}{640}$	0.00044857	*	0.000672855	*

Table 6
The maximum errors and convergence orders of numerical scheme when $\beta = 0.7$

h	$\underline{E}_{\infty}$(h)	Order1	$\overline{E}_{\infty}$(h)	Order2
$\frac{1}{80}$	0.0149139	1.63051	0.0223708	1.63051
$\frac{1}{160}$	0.0048168	1.65202	0.0072252	1.65202
$\frac{1}{320}$	0.00153268	1.66509	0.00229901	1.66509
$\frac{1}{640}$	0.000483289	*	0.000724934	*

the corresponding solution of the 1-system has necessarily the following expression

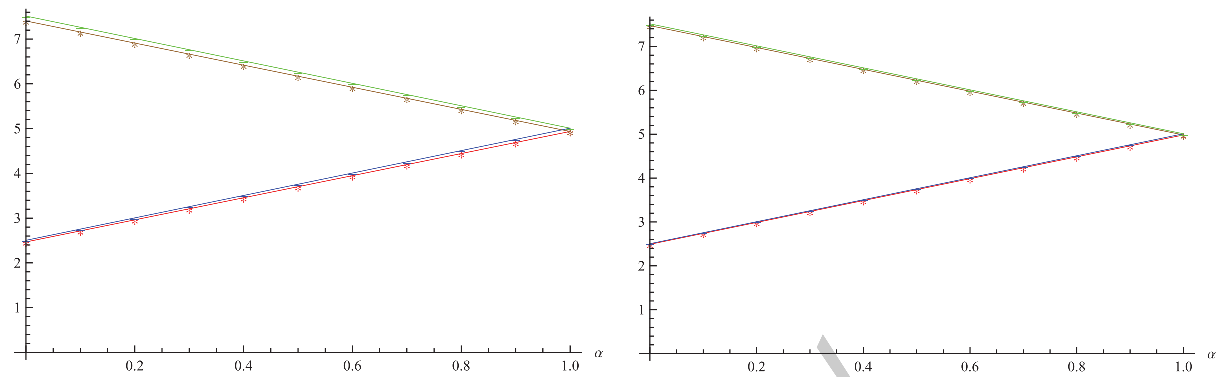
$$y(x) = (\underline{y}_0^{\alpha} E_{\beta}(x^{\beta}), \overline{y}_0^{\alpha} E_{\beta}(x^{\beta})), \tag{15}$$

where $E_{\beta}(z)$ is the Mittag-Leffler function of z and is defined by

$$E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}.$$

The error is defined as follows:

$$\underline{E}_{\infty}(h) = \max_{0 \leq j \leq m, 0 \leq \alpha \leq 1} |\underline{y}^{\alpha}(x_j) - \underline{y}_j^{\alpha}|,$$



(a) m = 20 (b) m = 40

Fig. 2. Exact (-)(Blue, Green) and approximate values of (*)(Red, Brown) of lower and upper values of y at $x = 1$.

Table 7
Approximate numerical values of $\underline{y}_j^{\alpha}$

x |α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.
0.1	1.07768	1.11408	1.15048	1.18689	1.22329	1.25969	1.2961	1.3325	1.3689	1.4053	1.44171
0.2	1.42625	1.45854	1.49084	1.52314	1.55543	1.58773	1.62002	1.65232	1.68462	1.71691	1.74921
0.3	1.75148	1.78118	1.81088	1.84059	1.87029	1.89999	1.92969	1.95939	1.98909	2.0188	2.0485
0.4	2.08148	2.10926	2.13704	2.16481	2.19259	2.22036	2.24814	2.27591	2.30369	2.33146	2.35924
0.5	2.42738	2.45363	2.47987	2.50612	2.53237	2.55861	2.58486	2.61111	2.63736	2.6636	2.68985
0.6	2.79612	2.82111	2.84609	2.87108	2.89606	2.92105	2.94603	2.97101	2.996	3.02098	3.04597
0.7	3.19319	3.21711	3.24102	3.26493	3.28885	3.31276	3.33668	3.36059	3.3845	3.40842	3.43233
0.8	3.62355	3.64653	3.66952	3.6925	3.71549	3.73848	3.76146	3.78445	3.80743	3.83042	3.8534
0.9	4.09201	4.11418	4.13635	4.15852	4.18069	4.20286	4.22503	4.2472	4.26937	4.29154	4.31371
1.0	4.60353	4.62498	4.64642	4.66786	4.68931	4.71075	4.7322	4.75364	4.77509	4.79653	4.81797

Table 8
Approximate numerical values of $\overline{y}_j^{\alpha}$

x |α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1.5	1.45	1.4	1.35	1.3	1.25	1.2	1.15	1.1	1.05	1.
0.1	1.80574	1.76933	1.73293	1.69653	1.66012	1.62372	1.58732	1.55092	1.51451	1.47811	1.44171
0.2	2.07217	2.03987	2.00758	1.97528	1.94299	1.91069	1.87839	1.8461	1.8138	1.7815	1.74921
0.3	2.34551	2.31581	2.28611	2.25641	2.22671	2.197	2.1673	2.1376	2.1079	2.0782	2.0485
0.4	2.637	2.60922	2.58144	2.55367	2.52589	2.49812	2.47034	2.44257	2.41479	2.38702	2.35924
0.5	2.95232	2.92608	2.89983	2.87358	2.84733	2.82109	2.79484	2.76859	2.74235	2.7161	2.68985
0.6	3.29582	3.27083	3.24585	3.22086	3.19588	3.17089	3.14591	3.12092	3.09594	3.07095	3.04597
0.7	3.67146	3.64755	3.62364	3.59972	3.57581	3.5519	3.52798	3.50407	3.48016	3.45624	3.43233
0.8	4.08326	4.06028	4.03729	4.0143	3.99132	3.96833	3.94535	3.92236	3.89938	3.87639	3.8534
0.9	4.53541	4.51324	4.49107	4.4689	4.44673	4.42456	4.40239	4.38022	4.35805	4.33588	4.31371
1.0	5.03242	5.01097	4.98953	4.96808	4.94664	4.92519	4.90375	4.88231	4.86086	4.83942	4.81797

and

$$\overline{E}_{\infty}(h) = \max_{0 \le j \le m, 0 \le \alpha \le 1} |\overline{y}^{\alpha}(x_j) - \overline{y}_j^{\alpha}|.$$

Let

$$\text{Order1} = \log_2 \left(\frac{\underline{E}_{\infty}(h)}{\underline{E}_{\infty}(h/2)} \right),$$

Table 9
Numerical values of the approximate solutions $\underline{y}_j^{\alpha}$

x|α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.
0.1	1.64262	1.75302	1.86165	1.96873	2.07445	2.17895	2.28236	2.38477	2.48628	2.58697	2.68689
0.2	2.9477	3.10509	3.25927	3.41065	3.55955	3.70624	3.85094	3.99384	4.1351	4.27486	4.41323
0.3	4.4984	4.70273	4.90246	5.09818	5.29034	5.47932	5.66544	5.84897	6.03014	6.20913	6.38613
0.4	6.34123	6.59554	6.8438	7.08676	7.32502	7.55908	7.78937	8.01621	8.23993	8.46077	8.67896
0.5	8.51594	8.82468	9.1258	9.42021	9.70869	9.99187	10.2703	10.5443	10.8144	11.0808	11.3438
0.6	11.0637	11.4324	11.7916	12.1427	12.4864	12.8236	13.1549	13.4809	13.8019	14.1185	14.4309
0.7	14.0295	14.4644	14.888	15.3017	15.7065	16.1035	16.4933	16.8767	17.2541	17.6261	17.9931
0.8	17.4628	17.9713	18.4663	18.9495	19.4221	19.8854	20.3401	20.7872	21.2271	21.6606	22.0881
0.9	21.4189	22.0091	22.5834	23.1438	23.6918	24.2287	24.7556	25.2733	25.7828	26.2845	26.7792
1	25.9588	26.6399	27.3025	27.9488	28.5805	29.1993	29.8064	30.4028	30.9894	31.567	32.1364

Table 10
Numerical values of the approximate solutions $\overline{y}_j^{\alpha}$

x|α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1.5	1.45	1.4	1.35	1.3	1.25	1.2	1.15	1.1	1.05	1.
0.1	3.65415	3.55954	3.46452	3.36909	3.27321	3.17686	3.08001	2.98262	2.88467	2.78611	2.68689
0.2	5.73889	5.61017	5.48072	5.3505	5.21948	5.08759	4.9548	4.82103	4.68623	4.55033	4.41323
0.3	8.07209	7.90907	7.74499	7.57979	7.41342	7.24581	7.07688	6.90654	6.73471	6.56128	6.38613
0.4	10.7492	10.5496	10.3486	10.1461	9.94202	9.7363	9.52882	9.31948	9.10815	8.8947	8.67896
0.5	13.8325	13.5931	13.3519	13.1088	12.8637	12.6164	12.367	12.1152	11.8608	11.6038	11.3438
0.6	17.3797	17.0965	16.8111	16.5233	16.2331	15.9403	15.6447	15.3461	15.0445	14.7395	14.4309
0.7	21.4504	21.1189	20.7846	20.4475	20.1074	19.7641	19.4175	19.0674	18.7134	18.3554	17.9931
0.8	26.109	25.7238	25.3354	24.9436	24.5483	24.1492	23.746	23.3387	22.9267	22.51	22.0881
0.9	31.4256	30.981	30.5326	30.0801	29.6234	29.1623	28.6964	28.2255	27.7492	27.2673	26.7792
1	37.4779	36.9673	36.4521	35.9323	35.4074	34.8774	34.3418	33.8003	33.2525	32.698	32.1364

Table 11
Approximate numerical values of $\underline{y}_j^{\alpha}(1 \leq j \leq 10)$

x |α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.
0.1	2.34265	2.37934	2.41542	2.45095	2.48597	2.52051	2.55461	2.58828	2.62154	2.65441	2.68689
0.2	4.12721	4.15817	4.18851	4.2183	4.24756	4.27632	4.3046	4.33243	4.3598	4.38674	4.41323
0.3	6.15394	6.18007	6.20543	6.23008	6.25409	6.27748	6.3003	6.32255	6.34427	6.36546	6.38613
0.4	8.4821	8.50522	8.52743	8.54881	8.56942	8.58931	8.60851	8.62706	8.64497	8.66227	8.67896
0.5	11.1757	11.1966	11.2163	11.2351	11.253	11.27	11.2863	11.3017	11.3165	11.3305	11.3438
0.6	14.2886	14.3074	14.3251	14.3416	14.3571	14.3716	14.3852	14.3979	14.4097	14.4207	14.4309
0.7	17.8753	17.8924	17.9082	17.9226	17.9358	17.9479	17.9589	17.9689	17.9779	17.986	17.9931
0.8	21.9945	22.01	22.0239	22.0363	22.0473	22.057	22.0655	22.0728	22.079	22.0841	22.0881
0.9	26.7103	26.7243	26.7363	26.7467	26.7554	26.7627	26.7686	26.7731	26.7764	26.7784	26.7792
1.0	32.0932	32.1056	32.1158	32.1241	32.1306	32.1353	32.1385	32.1401	32.1403	32.139	32.1364

Table 12
Approximate numerical values of $\overline{y}_j^{\alpha}(1 \leq j \leq 10)$

x |α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1.5	1.45	1.4	1.35	1.3	1.25	1.2	1.15	1.1	1.05	1.
0.1	2.98932	2.96116	2.93246	2.90325	2.87358	2.84347	2.81295	2.78202	2.75069	2.71898	2.68689
0.2	4.76098	4.72924	4.69664	4.66329	4.62925	4.59458	4.55934	4.52355	4.48725	4.45047	4.41323
0.3	6.70351	6.67499	6.64559	6.61539	6.58447	6.55288	6.52066	6.48785	6.45447	6.42056	6.38613
0.4	8.9866	8.95948	8.93138	8.90237	8.87255	8.84196	8.81066	8.77868	8.74606	8.71281	8.67896
0.5	11.6484	11.6221	11.5947	11.5663	11.5369	11.5067	11.4756	11.4438	11.4112	11.3778	11.3438
0.6	14.737	14.7112	14.6841	14.6558	14.6264	14.5961	14.5648	14.5326	14.4995	14.4656	14.4309
0.7	18.3043	18.2786	18.2515	18.2231	18.1934	18.1626	18.1307	18.0977	18.0638	18.0289	17.9931
0.8	22.4073	22.3817	22.3543	22.3254	22.2952	22.2636	22.2307	22.1967	22.1616	22.1254	22.0881
0.9	27.1093	27.0835	27.0557	27.0261	26.995	26.9624	26.9283	26.8929	26.8562	26.8183	26.7792
1.0	32.4803	32.454	32.4256	32.3951	32.3629	32.3289	32.2934	32.2563	32.2177	32.1778	32.1364

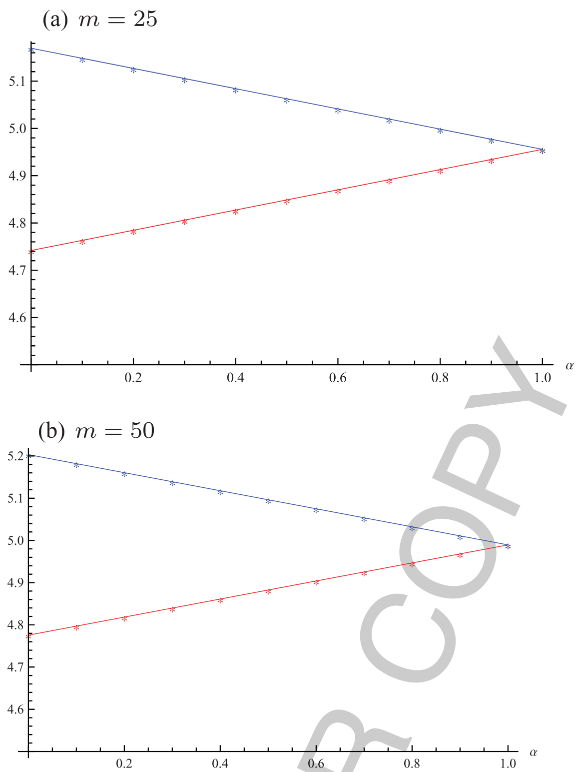


Fig. 3. Approximate solutions of $\underline{y}_m^\alpha$ (Red) and $\overline{y}_m^\alpha$ (Blue).

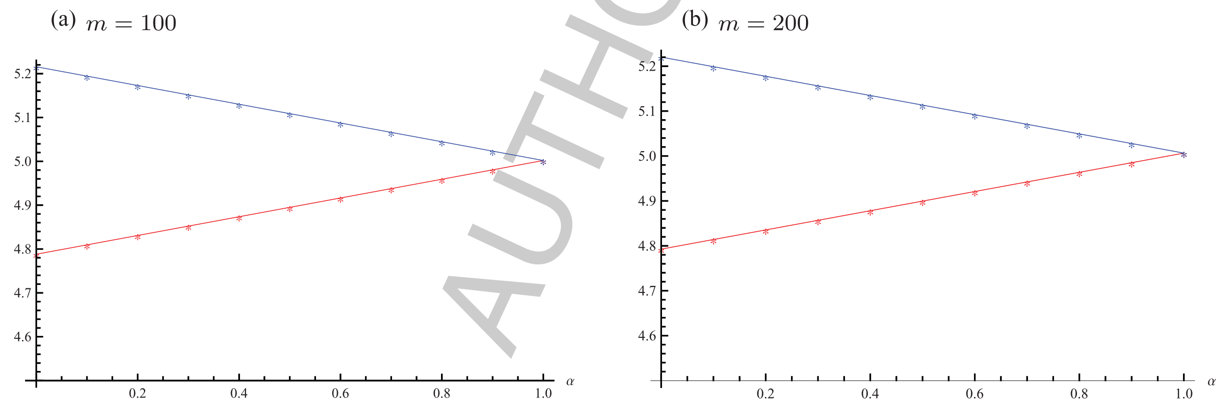


Fig. 4. Approximate solutions of $\underline{y}_m^\alpha$ (Red) and $\overline{y}_m^\alpha$ (Blue).

$$\text{Order2} = \log_2 \left[\frac{\overline{E}_\infty(h)}{\overline{E}_\infty(h/2)} \right].$$

We take $\beta = 1/2$ in Tables 1–5 and Fig. 1. The exact and approximate solutions $\underline{y}^\alpha(x_j)$, $\underline{y}_j^\alpha$ (blue and Red colour) and $\overline{y}^\alpha(x_j)$, $\overline{y}_j^\alpha$(green and brown colour) are plotted at $x_j = 1$ when $m = 10, m = 20$ and $m = 40$ respectively in Figs 1, 2(a) and 2(b).

From these figures, we see that the solutions are triangular fuzzy numbers. The numerical values of exact solutions $\underline{y}^\alpha(x_j)$ and $\overline{y}^\alpha(x_j)$ are given in Table 1 and Table 2 respectively. In Table 3 and Table 4, numerical values of the approximate solutions $\underline{y}_j^\alpha$ and $\overline{y}_j^\alpha$ are given respectively. In Table 5 and Table 6, we have compared the errors as well as their convergence order for the numerical scheme in the

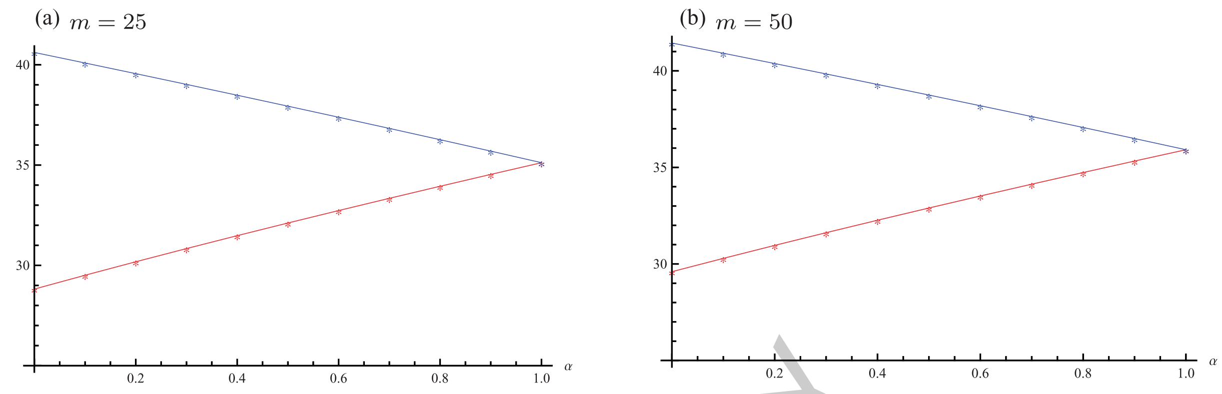


Fig. 5. Numerical solutions (*)(Red, Blue) of $\underline{y}_m^\alpha$ and $\overline{y}_m^\alpha$.

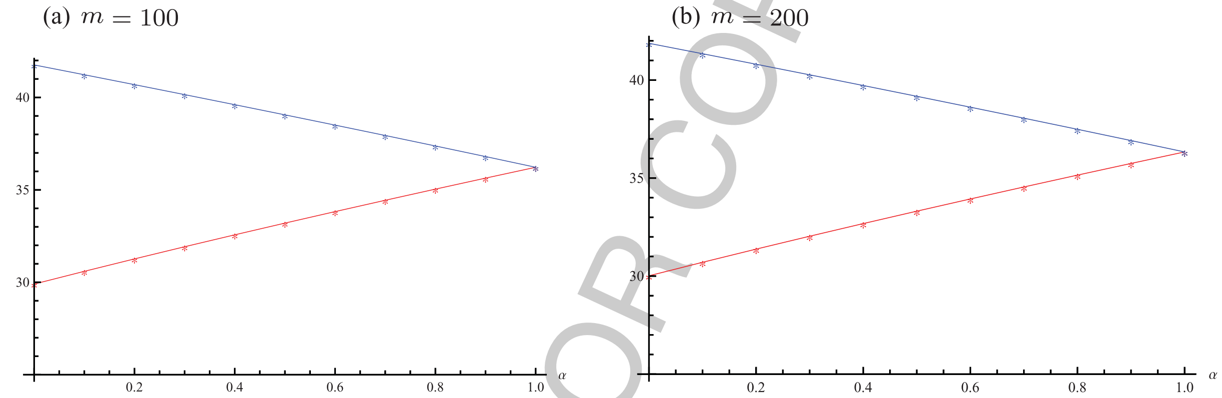


Fig. 6. Numerical solutions of $\underline{y}_m^\alpha$(Red) and $\overline{y}_m^\alpha$(Blue).

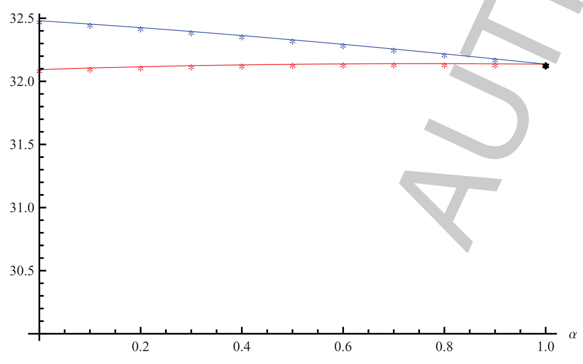


Fig. 7. Numerical solutions of $\underline{y}_m^\alpha$ (Red) and $\overline{y}_m^\alpha$ when $m = 10$(Blue).

case when h decreases for $\beta = 0.5$ and $\beta = 0.7$ respectively.

Case (ii) ${}_0^C D^\beta y$ is differentiable in second form. That is, $[{}_0^C D_2^\beta y(x)]^\alpha = [{}_0^C D^\beta \overline{y}^\alpha(x), {}_0^C D^\beta \underline{y}^\alpha(x)]$ and satisfies the 2-system associated with (14).

Numerical values of the approximate solutions $\underline{y}_j^\alpha$ and $\overline{y}_j^\alpha$ are tabulated in Table 7 and Table 8 respectively. Approximate solutions $\underline{y}_m^\alpha$(red colour) and $\overline{y}_m^\alpha$(blue colour) for different value of m are plotted in Figs. 3(a), 3(b), 4(a) and Fig. 4(b).

Example 2. We consider the nonlinear fuzzy fractional initial value problem of the form

$$\left.\begin{array}{l} {}_0^C D^{1/2} y(x) = \frac{9}{4}\sqrt{y(x)} + y(x), \\ y(0) = (0.5\alpha + 0.5, -0.5\alpha + 1.5), \quad x \in [0, 1], \end{array}\right\} \quad (16)$$

where $y(0) \in \mathbb{R}_F$.

Case (i) ${}_0^C D^\beta y$ is differentiable in first form. That is, $[{}_0^C D_1^\beta y(x)]^\alpha = [{}_0^C D^\beta \underline{y}^\alpha(x), {}_0^C D^\beta \overline{y}^\alpha(x)]$ and it satisfies the 1-system associated with (16). Numerical values of the approximate solutions $\underline{y}_j^\alpha$ and $\overline{y}_j^\alpha$ are tabulated in Table 9 and Table 10 respectively. Approximate solutions $\underline{y}_m^\alpha$(red colour) and $\overline{y}_m^\alpha$(blue colour) for different value of m are plotted in Figs. 5(a), 5(b), 6(a) and 6(b).

Case (ii) ${}_0^C D^\beta y$ is differentiable in second form. That is, $[{}_0^C D_2^\beta y(x)]^\alpha = [{}_0^C D^\beta \overline{y}^\alpha(x), {}_0^C D^\beta \underline{y}^\alpha(x)]$ and satisfies the 2-system associated with (14).

Numerical values of the approximate solutions $\underline{y}_j^\alpha$ and $\overline{y}_j^\alpha$ are tabulated in Table 11 and Table 12 respectively. Approximate solutions $\underline{y}_m^\alpha$ (red colour) and $\overline{y}_m^\alpha$ (blue colour) when $m = 10$ are plotted in Fig. 7.

6. Conclusion

The predictor-corrector method is used to solve fuzzy fractional initial value problems. The fractional product rectangle rule is used as a prediction at each step and the product trapezoidal rule is used to make correction to obtain the value at each step. Numerical solutions of fuzzy fractional initial value problems are compared with the corresponding exact solutions. The order of convergence of the predictor-corrector method is $\min\{2, 1 + \beta\}$.

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