β* - Continuous Maps and Pasting Lemma in Topological Spaces

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Abstract— In this paper, the authors introduce a new class of maps called β* continuous maps and β* irresolute maps in topological spaces and study some of its basic properties and relations among

Index Terms— g-closed, g-continuous,β * -closed,β * continuous, β * irresolute.

1.Introduction

Biswas[3], Husain[10], Ganster and Reilly[9], Levine[11,13], Marcus[15], Mashour [16]et al,Noiri[18], Noiri and Ahmed[19] and Tong[15,16,17] have introduced and investigated simple continuous, almost continuous, LC-continuity, weak continuity, semi-continuity,quasi-continuity, α-continuity, strong semi-continuity, semi-weak continuity, weak almost continuity, A-continuity and B-continuity respectively. Balachandran et al have introduced and studied generalized semi-continuous maps, semi locally continuous maps, semigeneralized locally con-tinuous maps and generalized locally continuous maps. Maki and Noiri studied the pasting lemma for α-continuous mappings. El Etik[7] also introduced the concept of *gb*-continuous functions with the aid of *b*-open sets,Omari and Noorani introduced and studied the concept of generalized g-closed sets and *g* -continuous maps in topological spaces. Palanimani and Parimelazhagan[26] introduced and studied theproperties of β*-closed set in topological spaces. Crossley and Hildebrand[5] introducedand investigated irresolute functions which are stronger than semi continuous maps but are independent of continuous maps. Since then several researchers have introduced several strong and weak forms of irresolute functions. Di Maio and Noiri[6], Faro[8], Cammaroto and Noiri [4], Maheswari and Prasad[16] and Sundaram [21] have introduced and studied quasi-irresolute and strongly irresolute maps strongly α-irresolute maps, almost irresolute maps, α -irresolute maps and gc-irresolute maps are respectively.The aim of this paper is to introduce and study the concepts of new class of maps namely β*-continuous maps and β*-irresolute maps.Throughout this paper (X,τ) and (Y,σ)(or simply X and Y) represents the non--empty topological spaces on which no separation axiom are assumed, un-less otherwise mentioned. For a subset A of X, cl(A) and int(A) represents the closure of A and interior of A respectively.

We recall the following definitions.

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2. Preliminaries

Definition 2.1[13] : A subset A of a topological space (X,τ) is called generalized closed set (briefly g-closed) if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X .

Definition 2.2 : A subset A of a topological space (X,τ) is called β* closed set. If $cl(int(A)) \subseteq U$, whenever $A \subseteq U$ and U is g-open in X .

Definition 2.3[25] : A subset A of a topological space (X,τ) is called g*-closed if $cl(A) \subseteq U$,whenever $A \subseteq U$ and U is g-open in X .

Definition 2.4: A map f : $(X,\tau) \to (Y,\sigma)$ from a topological space X into a topological space Y is called g-continuous if $f^{-1}(V)$ is g-closed in X for every closed set V of Y.

Definition 2.5: A map f : $(X,\tau) \to (Y,\sigma)$ from a topological space X into a topological space Y is called β*-continuous if $f^{-1}(V)$ is β*- closed in X for every closed set V of Y.

Definition 2.6: A map f : $(X,\tau) \to (Y,\sigma)$ from a topological space X into a topological space Y is called irresolute if $f^{-1}(V)$ is semi-closed in X for every semi-closed set V of Y.

Definition 2.7[5]: A map f : $(X,\tau) \to (Y,\sigma)$ from a topological space X into a topological space Y is called semi-generalized continuous(briefly sg continuous) if $f^{-1}(V)$ is sg-closed in X for every closed set V of Y.

3. β*- Continuous Maps

In this section we introduce the concept of β*-Continuous maps in topological spaces.

Definition 3.1 Let f : X → Y from a topological space X into a topological space Y is called β*-continuous if the inverse image of every closed set in Y is β*-closed in X.

Theorem 3.2 If a map f : X → Y from a topological space X into a topological space Y is continuous, then it is β*-continuous but not conversely.

Proof: Let f : X → Y be continuous. Let F be any closed set in Y. Then the inverse image $f^{-1}(F)$ is closed in X. Since every closed set is β*-closed, $f^{-1}(F)$ is β*-closed in X. Therefore f is β*-continuous.

Remark 3.3 The converse of the theorem 3.2 need not be true as seen from the following example

Example 3.4 : Let X = Y={a, b, c} with topologies τ = {X, ϕ, {a}, {a, b}} and σ = {Y, ϕ, {a},{b,c}}. Let f : X→ Y be a map defined by f (a) = c, f (b) = b, f (c) = a. Here f is β*-continuous. But f is

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not continuous since for the closed set F={b,c} in Y,$f^{-1}(F)$={b, a} is not closed in X.

Theorem 3.5: If a map f : X → Y from a topological space X into a topological space Y is g continuous,then it is β*-continuous but not conversely.

Proof: Let f : X → Y be g-continuous. Let F be any closed set in Y. Then the inverse image $f^{-1}(F)$ is g-closed in X. Since every g-closed set is β * -closed in X, $f^{-1}(F)$ is β * -closed in X. Therefore f is β * -continuous.

Remark 3.6 The converse of the theorem 3.5 need not be true as seen from the following example

Example 3.7 Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a}, {a,b}} and σ = {Y,ϕ,{a},{b}, {a,b}}.Let f : X → Y be the identity map. Here f is β * -continuous. But f is not g-continuous since for the closed set F={ a,c} f is not β * -closed set in X.

Theorem 3.8: If a map f : X → Y from a topological space X into a topological space Y is sg-continuous, then it is β *-continuous but not conversely.

Proof: Let f : X → Y be sg-continuous. Let V be any sg-closed set in Y. Then the inverse image $f^{-1}(V)$ is β * -closed in X. Since every β * -closed set is sg-closed, $f^{-1}(V)$ is β * -closed in X. Therefore f is β * -continuous.

Remark 3.9 The converse of the theorem 3.8 need not be true as seen from the following example.

Example 3.10 Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a}, {a,b}} and σ = {Y, ϕ,{b}, {a,c}}. Let f : X → Y be a map defined by f (a) = b, f (b) = a, f (c) =c. Here f is β * -continuous. But f is not sg-continuous since for the closed set F={ a,c} f is not β * -closed set in X.

Theorem 3.11: If a map f : X → Y from a topological space X into a topological space Y is gs-continuous, then it is β * -continuous but not conversely.

Proof: Let f : X → Y be sg-continuous. Let V be any gs-closed set in Y. Then the inverse image $f^{-1}(V)$ is β * -closed in X. Since every β * -closed set is gs-closed,$f^{-1}(V)$ is β *-closed in X. Therefore f is β * -continuous.

Remark 3.12 The converse of the theorem 3.11 need not be true as seen from the following example.

Example 3.13 Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a}, {a,b}} and σ = {Y,ϕ,{a},{c},{a,c}}. Let f : X → Y be a map defined by f (a) = b, f (b) = c, f (c) =b. Here f is β*-continuous. But f is not sg-continuous since for the closed set F={ a,b} f is not β*-closed set in X.

Theorem 3.14 If a map f : X → Y from a topological space X into a topological space Y

(i) The following statements are equivalent.

(a) f is β*-continuous.

(b) The inverse image of each open set in Y is β*-open in X.

(ii) If f : X → Y is β*-continuous, then f (β*(A)) ⊂ cl(int (f(A))) for every subset A of X.

(iii) The following statements are equivalent.

(a) For each point $x \in X$ and each open set V in Y with f (x) ∈ V,there is a β* -open set U in X such that x ∈ U , f (U) ⊂ V .

(b) For every subset A of X , f (β* (A)) ⊂cl(int (f(A))) holds.

(c) For each subset B of Y, β* (f^{-1}(B)) ⊂ f^{-1}(cl(int ((A)).

Proof: (i) Assume that f : X → Y be β*-continuous. Let G be open in Y.Then G^c is closed in Y . Since f is β*-continuous , $f^{-1}(G^c)$ is β*-closed in X. But $f^{-1}(G^c)$= X - $f^{-1}(G)$ .Thus X - $f^{-1}(G)$ is β*- closed in X and so $f^{-1}(G)$ is β*-open in X. Therefore

(a) implies (b).

Conversely assume that the inverse image of each open set in Y is β*-open in X. Let F be any closed set in Y. The F^c is open in Y. By assumption, $f^{-1}(F^c)$ is β*-open in X. But $f^{-1}(F^c)$ = X – $f^{-1}(F)$. Thus X – $f^{-1}(F)$ is β *-open in X and so $f^{-1}(F)$ is β*-closed in X. Therefore f is β*-continuous.

Hence (b) implies (a). Thus (a) and (b) are equivalent.

(ii) Assume that f is β*-continuous. Let A be any subset of X. Then cl(int(f (A)))is closed in Y. Since f is β*-continuous , $f^{-1}\left(cl\left(int\left(f(A)\right)\right)\right)$ is β*-closed in X and it contains A. Therefore β*(A)⊂ $f^{-1}\left(cl\left(int\left(f(A)\right)\right)\right)$ and so $f(\beta^*(A) \subset cl\left(int\left(f(A)\right)\right)$.

(a) ⇒ (b).

Let y ∈ β*(A) and $x \in X$, $f(x) \in V$. Let V be any neighborhood of y. Then there exists a point $x \in X$ and a β^*-open set U such that $f(x) = y$, x ∈ U , $x \in \beta^*(A)$,f (U) ∈ V .Since $x \in \beta^*(A)$,U∩$A \neq \varphi$ holds and hence $f(A) \cap V = \varphi$ .Therefore we have $y = f(x) \in cl(int(A)$.

(b) ⇒ (a).

Let $x \in X$ and V be a open set containing f (x) .Let $A = f^{-1}(V^c)$, then $x \notin A$.Since$f(\beta^*(A)) \subset cl(f(A)) \subset V^c$.Then $x \notin \beta^*(A)$, there exist a $\beta^*(A)$ open set U containing x such that U ∩A ∈ ϕ and hence f (U) ⊂ f (A^c) ⊂ V .

(b) ⇒ (c)

Let B be any subset of Y. Replacing A by $f^{-1}(B)$ we get from (b),$f\left(\beta^*(f^{-1}(B))\right) \subseteq cl\left(int(f^{-1}(B))\right) \subseteq B$. Hence $\beta^*(f^{-1}(B)) \subset f^{-1}\left(cl(int(B))\right)$.

(c) ⇒ (b)

Let B = f (A) where A is a subset of X. Then $\beta^*(A) \subset \beta^*(f^{-1}(B)) \subset f^{-1}\left(cl(int(A))\right)$Therefore $f(\beta^*(A)) \subset cl\left(int(f(A))\right)$

Remark 3.15 The converse of the theorem 3.14(ii) need not be true as seen from the following example.

Example 3.16 Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a}, {a,b}} and σ = {Y,ϕ,{a},{b}, {a,b}}.Let f : X → Y be the identity map. Here f is β*-continuous. But f is not g-continuous since for the closed set F={ a,c} f is not β * -closed set in X.Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a}, {a,b}} and σ = {Y,ϕ,{a},{c}, {a,c}}. f : X → Y be a map defined by f (a) = a, f (b) = b, f (c) = c. then for every subset A of X $f(\beta^*(A)) \subset cl(int(A))$ holds but it is not β*-continuous since for a closed set {a,c} in Y $f^{-1}(\{a,c\}) = \{a,b\}$ is not β*-closed in X .

Theorem 3.16 If f : X → Y and g :Y → Z be any two functions, Then $g \circ f: X \to Z$ is β*-continuous if g is continuous and f is β*-continuous

Proof: Let V be a closed set in Z, Since g is continuous, $g^{-1}(V)$ is closed in V. and since f is β * continuous.$f^{-1}(g^{-1}(V))$ is β*-closed in X . thus g◎ f is β * continuous.

Remark 3.17 The composition of two β*-continuous map need not be β*- continuous. Let us prove the remark by the following example.

Example 3.18 Let X = Y = Z = {a, b, c} with topologies τ = {X, ϕ, {a}, {a, b}}, σ = {Y, ϕ,{a},{c}, {a, c}} and η = {Z, ϕ, {a, c}}. Let g : (X, τ) →(Y, σ) be a map defined by g(a) = a, g(b) = c and g(c) = b ,let f : (Z, η) → (X, τ) be a map defined by f (a) = a, f (b) = c and f(c) = b. Both f and g are β*-continuous.

Define g ∘ f : (Z, η) → (Y, σ). Here {b} is a closed set of (Y, σ). Therefore $(g \circ f)^{-1}(\{b\}) = \{c\}$ is not a β *-closed set of (Z, η).

Hence g ∘ f is not β*-continuous.

4 β *- Irresolute Maps

Definition 4.1 A map f : X → Y from a topological space Y is called β*-irresolute if the inverse image of every β*-closed set in Y is β*-closed in X.

Theorem 4.2 A map f : X → Y is β * -irresolute if and only if the inverse image of every β * -open set in Y is β * -open in X.

Proof: Assume that f is β*-irresolute. Let A be any β*-open set in Y. Then A^c is β*-closed set in Y. Since f is β*-irresolute, $f^{-1}(A^c)$ is β*-closed in X.But $f^{-1}(A^c) = X - f^{-1}(A)$ and so $f^{-1}(A)$ is β* -open in X. Hence the inverse image of every β*-open set in Y is β*-open in X. Conversely assume that the inverse image of every β * -open set in Y is β*-open in X. Let A be any β*-closed set in Y. Then A^c is β*-open in Y. By assumption, $f^{-1}(A^c)$ is β*-open in X. But $f^{-1}(A^c) = X - f^{-1}(A)$ and so $f^{-1}(A)$ is β *-closed in X. Therefore f is β *-irresolute.

Theorem 4.3 If a map f : X → Y is β*-irresolute, then it is β*-continuous but not conversely.

Proof: Assume that f is β*-irresolute. Let F be any closed set in Y. Since every closed set is β*-closed, F is β*-closed in Y. Since f is β*-irresolute, $f^{-1}(F)$ is β*-closed in X. Therefore f is β*-continuous.

Remark 4.4 The converse of the theorem 4.3 need not be true as seen from the following example.

Example 4.5 Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a}, {a,b}} and σ = {Y,ϕ,{a},{c}, {a,c}} Let f : (X, τ) → (Y, σ) be map defined by f (a) = a, f (b) = c and f (c) = b. . Here f is β*- continuous. But {a, c} is β *-closed in Y but $f^{-1}(\{a, c\}) = \{a, b\}$ is not β*-closed in X. Therefore f is not β*-irresolute.

Theorem 4.6 Let X, Y and Z be any topological spaces. For any β*- irresolute map f : X → Y and any β*-continuous map g : Y → Z, the composition g∘f :(X, τ) → (Z, η) is β*-continuous.

Proof: Let F be any β*-closed set in Z. Since g is β*- continuous , $g^{-1}(F)$ is β*-closed in Y. Since f is β*-irresolute, $f^{-1}(g^{-1}(F))$ is β*-closed in X. But $f^{-1}(g^{-1}(F))=(g \circ f)^{-1}(F)$. Therefore $g \circ f : X \to Z$ is β*-continuous.

Remark 4.7 The irresolute maps and β*-irresolute maps are independent of each other. Let us prove the remark by the following two examples.

Example 4.8 Let X = Y = {a, b, c} with topologies τ= {X, ϕ, {a}, {b}, {a, b}} and σ = {Y, ϕ, {b}, {b, c}}. Then the identity map f : (X, τ) → (Y, σ) is irresolute, but it is not β*-irresolute. Since G = {a,c} is β*-closed in (Y, σ),where $f^{-1}(G) = \{a,c\}$ is not β*-closed in (X, τ).

Example 4.9 Let X = Y = {a, b, c} with topologies τ = {X, ϕ, {a, b}} and σ = {Y, ϕ, {a}}. Then the identity map f : (X, τ) → (Y, σ) is β *-irresolute, but it is not irresolute. Since G = {a, c} is semi-open in (Y, σ), where $f^{-1}(G) = \{a,c\}$ is not semi-open in (X, τ).

There fore it is evident that

$$\text{irresolute} \underset{\nrightarrow}{\overset{\nleftarrow}{-}} \beta^* \text{ irresolute} \underset{\rightarrow}{\overset{\nleftarrow}{-}} \beta^* \text{ continous}$$

5 Pasting Lemma for β * -Continuous Maps

Theorem 5.1 Let X = A ∪ B be a topological space with topology τ and Y be a topological space with topology σ.

Let f : (A, τ/A) → (Y, σ) and g :(B, τ/B) → (Y, σ) be β *- continuous maps such that f(x) = g(x) for every x ◎A ∩ B. Suppose that A and B are β *-closed sets in X. Then the combination α : (X, τ) → (Y, σ) is β *-continuos.

Proof: Let F be any closed set in Y. Clearly $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$= C ∪ D where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. But C is β*-closed in A and A is β *-closed in X and so C is β*-closed in X. Since we have proved that if B◎A◎ X, B is β *-closed in A and A is β*-closed in X then B is β*-closed in X. Also C ∪ D is β*-closed in X. Therefore $\alpha^{-1}(F)$ is β*-closed in X. Hence α is β*-continous.

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